Fourier Series and Laplace Transforms
Solutions Manual

Contents

Chapter 1, Section 1
Linear Spaces and Inner Products

Unless stated otherwise, all functions are real-valued and all function spaces are over $\mathbb{R}$.

Exercise 1: Intersections and unions of vector spaces

Let $V_1, V_2, \ldots, V_n$ be vector spaces over $\mathbb{C}$. Prove that

$$V = \bigcap_{k=1}^{n} V_k$$

is a vector space. What can be said about

$$W = \bigcup_{k=1}^{n} V_k?$$

Solution. All the spaces are understood to be subspaces of one common ambient vector space.

First, $0 \in V_k$ for every k, hence $0 \in V$. If $x, y \in V$, then $x, y \in V_k$ for every k. Since each V_k is a vector space,

$$x + y \in V_k \qquad (k = 1, \ldots, n),$$

and therefore $x + y \in V$. Likewise, for every $\lambda \in \mathbb{C}$,

$$\lambda x \in V_k \qquad (k = 1, \ldots, n),$$

so $\lambda x \in V$. Thus V is a vector space.

The union W need not be a vector space. For example, in $\mathbb{R}^2$ let

$$V_1 = \{(x, 0) : x \in \mathbb{R}\}, \qquad V_2 = \{(0, y) : y \in \mathbb{R}\}.$$

Then $(1, 0), (0, 1) \in V_1 \cup V_2$, but

$$(1, 0) + (0, 1) = (1, 1) \notin V_1 \cup V_2.$$

For a finite family, the precise statement is:

$$\boxed{\bigcup_{k=1}^{n} V_k \text{ is a vector space if and only if one } V_j \text{ contains all the others.}}$$

The "if" direction is immediate. For the converse, suppose that W is a vector space and that none of the V_j equals W. After removing redundant members, we may assume that no V_i is contained in another V_j.

We use the following elementary fact: a vector space over an infinite field cannot be the union of finitely many proper subspaces. Indeed, choose

$$x \in V_n \setminus \bigcup_{i<n} V_i$$

and choose $y \in V_1 \setminus V_n$. Such choices are possible after deleting redundant subspaces (or by induction on the number of subspaces). Consider the affine line

$$L = \{x + \lambda y : \lambda \in \mathbb{C}\}.$$

For each fixed i, the line L meets V_i in at most one point: if two distinct points $x + \lambda y$ and $x + \mu y$ belonged to V_i, then subtraction would give $y \in V_i$, and then also $x \in V_i$, contrary to the way x or y was chosen. Since $\mathbb{C}$ is infinite, L has infinitely many points, while finitely many subspaces can cover at most finitely many points of L. This contradiction proves that one of the V_j must equal W, hence it contains all the others.

Exercise 2: Absolutely integrable functions

Prove that

$$V = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is absolutely integrable on } \mathbb{R}\}$$

is a vector space over $\mathbb{R}$.

Solution. The zero function belongs to V, since

$$\int_{-\infty}^{\infty} |0| \, dx = 0.$$

Let $f, g \in V$ and let $\alpha, \beta \in \mathbb{R}$. By the triangle inequality,

$$|\alpha f(x) + \beta g(x)| \le |\alpha| \, |f(x)| + |\beta| \, |g(x)|.$$

Therefore

$$\int_{-\infty}^{\infty} |\alpha f(x) + \beta g(x)| \, dx \le |\alpha| \int_{-\infty}^{\infty} |f(x)| \, dx + |\beta| \int_{-\infty}^{\infty} |g(x)| \, dx < \infty.$$

Thus $\alpha f + \beta g \in V$, and V is a vector space over $\mathbb{R}$.

Exercise 3: Which formulas define inner products?

Let $C[-1, 2]$ denote the real vector space of continuous functions on $[-1, 2]$. Determine which of the following expressions define inner products.

(a) $\langle f, g\rangle = \displaystyle\int_{-1}^{2} |f(t) + g(t)| \, dt.$

(b) $\langle f, g\rangle = \displaystyle\int_{-1}^{2} f(t)g(t) \, dt + f\left(-\frac{1}{2}\right) g\left(-\frac{1}{2}\right).$

(c) $\langle f, g\rangle = 3\displaystyle\int_{-1}^{2} f(t)g(t) \, dt.$

(d) $\langle f, g\rangle = f(0)g(0) + f(1)g(1).$

Solution. **(a) Not an inner product.** It is not linear in either variable. For example,

$$\langle f, 0\rangle = \int_{-1}^{2} |f(t)|\, dt,$$

which is generally nonzero, whereas an inner product must satisfy $\langle f, 0\rangle = 0$.

(b) Yes. Symmetry and bilinearity are clear. Moreover,

$$\langle f, f\rangle = \int_{-1}^{2} f(t)^2\, dt + f\Big(-\frac{1}{2}\Big)^2 \geq 0.$$

If $\langle f, f\rangle = 0$, then the nonnegative continuous function f^2 has integral zero, hence $f(t) = 0$ for all $t \in [-1, 2]$. Therefore the form is positive definite.

(c) Yes. This is three times the usual L^2 inner product on continuous functions:

$$\langle f, f\rangle = 3\int_{-1}^{2} f(t)^2\, dt.$$

It is bilinear, symmetric, and positive definite.

(d) Not an inner product. It is only positive semidefinite. A nonzero continuous function may vanish at both 0 and 1. For example,

$$f(t) = t(t-1)$$

is nonzero but satisfies

$$\langle f, f\rangle = f(0)^2 + f(1)^2 = 0.$$

Thus positive definiteness fails.

Exercise 4: A form involving second derivatives

Let V be the real vector space of twice differentiable functions on $[-\pi, \pi]$. Define

$$\langle f, g\rangle = f(-\pi)g(-\pi) + \int_{-\pi}^{\pi} f''(x)g''(x)\, dx.$$

Is this an inner product on V?

Solution. The form is symmetric and bilinear, and $\langle f, f\rangle \geq 0$. However, it is not positive definite.

Take

$$f(x) = x + \pi.$$

Then f is not the zero function, but

$$f(-\pi) = 0, \qquad f''(x) = 0.$$

Hence

$$\langle f, f\rangle = 0.$$

Therefore the proposed form is not an inner product on V.

Exercise 5: Point-evaluation data on $C^1[0,1]$

Let $C^1[0,1]$ be the real vector space of continuously differentiable functions on $[0,1]$. Define

$$\langle f, g\rangle = f(0)g(0) + f'(0)g'(0) + f(1)g(1).$$

(a) Is this an inner product on the subspace

$$P_2 = \operatorname{span}\{1, x, x^2\}?$$

(b) Is it an inner product on all of $C^1[0,1]$?

Solution. **(a) Yes.** Bilinearity and symmetry are immediate. Let

$$p(x) = a + bx + cx^2 \in P_2.$$

If $\langle p, p\rangle = 0$, then

$$p(0) = 0, \qquad p'(0) = 0, \qquad p(1) = 0.$$

These conditions give

$$a = 0, \qquad b = 0, \qquad a + b + c = 0,$$

so $c = 0$. Hence $p \equiv 0$, proving positive definiteness on P_2.

(b) No. On the full space, the three measurements $f(0)$, $f'(0)$, and $f(1)$ do not determine the function. For example,

$$f(x) = x^2(1-x)^2$$

is nonzero and belongs to $C^1[0,1]$, but

$$f(0) = 0, \qquad f'(0) = 0, \qquad f(1) = 0.$$

Thus $\langle f, f\rangle = 0$ although $f \not\equiv 0$.

Exercise 6: Three more candidate inner products

Let $C^1[0,1]$ be as above. Determine which of the following formulas define inner products on $C^1[0,1]$.

(a) $\langle f, g\rangle = f(0)g(0) + \displaystyle\int_0^1 f'(t)g'(t)\,dt.$

(b) $\langle f, g\rangle = f(0)g(0) + f'(1)g'(1).$

(c) $\langle f, g\rangle = 2\displaystyle\int_0^{1/2} f(t)g(t)\,dt - \int_{1/2}^1 f(t)g(t)\,dt.$

Solution. **(a) Yes.** Symmetry and bilinearity are clear. Also,

$$\langle f, f\rangle = f(0)^2 + \int_0^1 |f'(t)|^2\,dt \ge 0.$$

If $\langle f, f\rangle = 0$, then $f(0) = 0$ and $f'(t) = 0$ for all t. Hence f is constant, and since $f(0) = 0$, we get $f \equiv 0$. Thus the form is positive definite.

(b) No. Positive definiteness fails. For example,

$$f(x) = x(2 - x)$$

is nonzero, while

$$f(0) = 0, \qquad f'(1) = 0.$$

Therefore $\langle f, f \rangle = 0$.

(c) No. The form is not even nonnegative. Choose any nonzero continuous function supported in $(1/2, 1]$. For instance, one may take a smooth function that vanishes on $[0, 1/2]$ but is not identically zero on $(1/2, 1]$. Then

$$\langle f, f \rangle = -\int_{1/2}^{1} f(t)^2 \, dt < 0.$$

Hence this expression cannot be an inner product.

Summary

Exercise / part	Result
3(a)	Not an inner product
3(b)	Inner product
3(c)	Inner product
3(d)	Not an inner product
4	Not an inner product
5(a)	Inner product on P_2
5(b)	Not an inner product on $C^1[0, 1]$
6(a)	Inner product
6(b)	Not an inner product
6(c)	Not an inner product

Chapter 1, Section 2
Norms and Inner Products

Convention: In complex inner-product spaces we take the inner product to be linear in the first variable and conjugate-linear in the second. Thus

$$\langle \alpha u, v\rangle = \alpha \langle u, v\rangle, \qquad \langle u, \alpha v\rangle = \overline{\alpha} \langle u, v\rangle .$$

Editorial note on Exercise 3(a). The standard equality statement in Cauchy–Schwarz is $|\langle u, v\rangle| = \|u\| \|v\|$, not $\langle u, v\rangle = \|u\| \|v\|$. The former is equivalent to complex linear dependence. The latter is stronger and forces the proportionality constant to be a nonnegative real number. We solve the standard intended formulation and record the stronger conclusion as well.

Exercise 1: Two norms on $\mathbb{R}^n$ and $\mathbb{C}^n$

Let $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$.

(a) The infinity norm

For $x = (x_1, \ldots, x_n) \in V$, define

$$\|x\|_\infty = \max_{1\le k\le n} |x_k|.$$

We verify the norm axioms.

1. *Positivity and definiteness.* Clearly $\|x\|_\infty \ge 0$. Moreover,

$$\|x\|_\infty = 0 \iff |x_k| = 0 \text{ for every } k \iff x = 0.$$

2. *Absolute homogeneity.* For every scalar α,

$$\|\alpha x\|_\infty = \max_k |\alpha x_k| = |\alpha| \max_k |x_k| = |\alpha| \|x\|_\infty .$$

3. *Triangle inequality.* For every k,

$$|x_k + y_k| \le |x_k| + |y_k| \le \|x\|_\infty + \|y\|_\infty .$$

Taking the maximum over k gives

$$\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty .$$

Thus $\|\cdot\|_\infty$ is a norm, called the infinity norm or uniform norm.

(b) The ℓ^1-norm

Define

$$\|x\|_1 = \sum_{k=1}^{n} |x_k|.$$

Positivity and definiteness are immediate, and

$$\|\alpha x\|_1 = \sum_{k=1}^{n} |\alpha x_k| = |\alpha| \sum_{k=1}^{n} |x_k| = |\alpha| \, \|x\|_1 \, .$$

Finally,

$$\|x+y\|_1 = \sum_{k=1}^{n} |x_k + y_k| \le \sum_{k=1}^{n} (|x_k| + |y_k|) = \|x\|_1 + \|y\|_1 \, .$$

Hence $\|\cdot\|_1$ is also a norm on V.

Exercise 2: Lagrange's identity for functions

Let $f, g \in C[a,b]$, with

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

We prove

$$\frac{1}{2}\int_a^b \int_a^b |f(x)g(y) - g(x)f(y)|^2 \, dx \, dy = \|f\|^2 \, \|g\|^2 - |\, \langle f, g \rangle \,|^2.$$

Expand the square:

$$\begin{aligned} |f(x)g(y) - g(x)f(y)|^2 = {} & |f(x)|^2 |g(y)|^2 + |g(x)|^2 |f(y)|^2 \\ & - f(x)g(y)\overline{g(x)f(y)} - g(x)f(y)\overline{f(x)g(y)}. \end{aligned}$$

After integration over $[a,b]^2$, the first two terms each give $\|f\|^2 \, \|g\|^2$. The third gives

$$\left(\int_a^b f(x)\overline{g(x)}\, dx \right) \left(\int_a^b g(y)\overline{f(y)}\, dy \right) = |\, \langle f, g \rangle \,|^2,$$

and the fourth gives the same quantity. Therefore

$$\int_a^b \int_a^b |f(x)g(y) - g(x)f(y)|^2 \, dx \, dy = 2 \, \|f\|^2 \, \|g\|^2 - 2|\, \langle f, g \rangle \,|^2,$$

which proves the identity.

Since the left-hand side is nonnegative,

$$\|f\|^2 \, \|g\|^2 - |\, \langle f, g \rangle \,|^2 \ge 0.$$

Hence

$$\boxed{|\, \langle f, g \rangle \,| \le \|f\| \, \|g\|},$$

which is the Cauchy–Schwarz inequality in $C[a,b]$.

Exercise 3: Equality cases

Let V be an inner-product space and let $u, v \neq 0$.

(a) Equality in Cauchy–Schwarz

We show
$$|\langle u, v\rangle| = \|u\|\,\|v\| \iff u = av \text{ for some } a \in \mathbb{C}.$$
If $u = av$, then
$$|\langle u, v\rangle| = |a|\,\|v\|^2 = \|u\|\,\|v\|.$$
Conversely, define
$$a = \frac{\langle u, v\rangle}{\|v\|^2}.$$
Then
$$\begin{aligned}\|u - av\|^2 &= \|u\|^2 - a\langle v, u\rangle - \overline{a}\langle u, v\rangle + |a|^2\|v\|^2 \\ &= \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2}.\end{aligned}$$
Under equality in Cauchy–Schwarz this is zero, so $u = av$.

If the sheet's stronger equality
$$\langle u, v\rangle = \|u\|\,\|v\|$$
is used literally, then the same argument yields $u = av$, and the displayed equality forces $a \geq 0$ real.

(b) Equality in the triangle inequality

We prove
$$\|u + v\| = \|u\| + \|v\| \iff u = av \text{ for some } a \geq 0.$$
Squaring the equality gives
$$\|u\|^2 + \|v\|^2 + 2\Re\langle u, v\rangle = \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\|,$$
so
$$\Re\langle u, v\rangle = \|u\|\,\|v\|.$$
But
$$\Re\langle u, v\rangle \leq |\langle u, v\rangle| \leq \|u\|\,\|v\|.$$
Thus equality holds throughout. Part (a) gives $u = av$, and the fact that $\langle u, v\rangle = a\|v\|^2$ is positive real implies $a \geq 0$.

Conversely, if $u = av$ with $a \geq 0$, then
$$\|u + v\| = \|(a+1)v\| = (a+1)\|v\| = \|u\| + \|v\|.$$

Exercise 4: The parallelogram law

For $u, v \in V$,
$$\begin{aligned}\|u + v\|^2 &= \langle u + v, u + v\rangle = \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \langle v, u\rangle, \\ \|u - v\|^2 &= \langle u - v, u - v\rangle = \|u\|^2 + \|v\|^2 - \langle u, v\rangle - \langle v, u\rangle.\end{aligned}$$
Adding gives
$$\boxed{\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2}.$$

Exercise 5: Real polarization identity

Assume V is a real inner-product space. Then

$$\|u+v\|^2 = \|u\|^2 + 2\langle u,v\rangle + \|v\|^2,$$
$$\|u-v\|^2 = \|u\|^2 - 2\langle u,v\rangle + \|v\|^2.$$

Subtracting,

$$\|u+v\|^2 - \|u-v\|^2 = 4\langle u,v\rangle.$$

Therefore

$$\boxed{\langle u,v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2}.$$

Exercise 6: Complex polarization identity

Assume V is a complex inner-product space, linear in the first variable. First,

$$\|u+v\|^2 - \|u-v\|^2 = 4\Re\langle u,v\rangle.$$

Also,

$$\|u+iv\|^2 = \|u\|^2 + \|v\|^2 + 2\Im\langle u,v\rangle,$$
$$\|u-iv\|^2 = \|u\|^2 + \|v\|^2 - 2\Im\langle u,v\rangle.$$

Thus

$$\|u+iv\|^2 - \|u-iv\|^2 = 4\Im\langle u,v\rangle.$$

Combining real and imaginary parts yields

$$\boxed{\langle u,v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 + \frac{i}{4}\|u+iv\|^2 - \frac{i}{4}\|u-iv\|^2}.$$

Exercise 7: Recovering an inner product from a norm

This is the Jordan–von Neumann theorem. Suppose a normed space $(V,\|\cdot\|)$ satisfies the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

We show that the norm comes from an inner product.

Real case

Define

$$\langle x,y\rangle := \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

Symmetry is immediate. Also,

$$\langle x,x\rangle = \|x\|^2,$$

so positivity and definiteness hold.

It remains to prove linearity. Put $Q(x) = \|x\|^2$ and

$$B(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y)).$$

The parallelogram identity implies the Jensen relation

$$B(x+z,y)+B(x-z,y)=2B(x,y).$$

Replacing x successively by suitable sums and differences gives

$$B(x+z,y)=B(x,y)+B(z,y).$$

Hence B is additive in its first variable. It follows that $B(mx,y)=mB(x,y)$ for integers m, then for rational scalars, and finally for real scalars by continuity of the norm. Symmetry gives linearity in the second variable as well. Therefore B is an inner product, and

$$B(x,x)=\|x\|^2.$$

Complex case

Define the complex polarization expression

$$\langle x,y\rangle := \frac{1}{4}(\|x+y\|^2-\|x-y\|^2)+\frac{i}{4}(\|x+iy\|^2-\|x-iy\|^2).$$

The same parallelogram-law argument proves additivity and real homogeneity. Direct substitution gives

$$\langle ix,y\rangle = i\langle x,y\rangle,$$

so the form is complex-linear in the first variable. Conjugate symmetry follows from the formula, and

$$\langle x,x\rangle=\|x\|^2\geq 0.$$

Thus this is an inner product inducing the original norm.

Hence a norm is induced by an inner product if and only if it satisfies the parallelogram law.

Exercise 8: The inequality of means

For real numbers $x_1,\ldots,x_n$, prove

$$\left|\frac{1}{n}\sum_{k=1}^n x_k\right| \leq \left(\frac{1}{n}\sum_{k=1}^n x_k^2\right)^{1/2}.$$

Apply Cauchy–Schwarz in $\mathbb{R}^n$ to

$$x=(x_1,\ldots,x_n),\qquad \mathbf{1}=(1,\ldots,1).$$

Then

$$\left|\sum_{k=1}^n x_k\right| = |\langle x,\mathbf{1}\rangle| \leq \|x\|_2\,\|\mathbf{1}\|_2=\left(\sum_{k=1}^n x_k^2\right)^{1/2}\sqrt{n}.$$

Divide by n:

$$\left|\frac{1}{n}\sum_{k=1}^n x_k\right| \leq \frac{1}{\sqrt{n}}\left(\sum_{k=1}^n x_k^2\right)^{1/2}=\left(\frac{1}{n}\sum_{k=1}^n x_k^2\right)^{1/2}.$$

Equality holds exactly when $(x_1,\ldots,x_n)$ is a scalar multiple of $(1,\ldots,1)$, that is, when all x_k are equal.

Chapter 1, Section 3
Orthogonal Systems and Gram–Schmidt

Exercise 1

Let $C[-1,1]$ be the complex vector space of continuous functions $f : [-1,1] \to \mathbb{C}$, equipped with

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,dx.$$

Define

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = 1 - 3x^2.$$

(a) Prove that these polynomials are mutually orthogonal.

(b) Find constants a, b, c such that

$$P_3(x) = a + bx + cx^2 + x^3$$

is orthogonal to each of P_0, P_1, P_2.

Solution

Since the displayed polynomials are real-valued, complex conjugation does not alter them.
First,

$$\langle P_0, P_1\rangle = \int_{-1}^{1} x\,dx = 0,$$

because the integrand is odd. Similarly,

$$\langle P_1, P_2\rangle = \int_{-1}^{1} x(1 - 3x^2)\,dx = 0.$$

Finally,

$$\langle P_0, P_2\rangle = \int_{-1}^{1} (1 - 3x^2)\,dx = 2 - 3 \cdot \frac{2}{3} = 0.$$

Thus P_0, P_1, P_2 are mutually orthogonal.
For part (b), impose the three orthogonality conditions. From $\langle P_3, P_0\rangle = 0$,

$$2a + \frac{2c}{3} = 0, \qquad extso \qquad a = -\frac{c}{3}.$$

From $\langle P_3, P_1\rangle = 0$, only the even integrand terms survive:

$$\int_{-1}^{1} (bx^2 + x^4)\,dx = \frac{2b}{3} + \frac{2}{5} = 0,$$

so
$$b = -\frac{3}{5}.$$
For $\langle P_3, P_2\rangle = 0$, the odd part of P_3 contributes nothing. Moreover, P_2 is already orthogonal to the constant polynomial, hence
$$0 = \langle P_3, P_2\rangle = c\int_{-1}^{1} x^2(1-3x^2)\,dx = c\left(\frac{2}{3}-\frac{6}{5}\right) = -\frac{8c}{15}.$$
Therefore $c = 0$, and consequently $a = 0$. Hence
$$\boxed{P_3(x) = x^3 - \frac{3}{5}x.}$$
Any nonzero scalar multiple of this polynomial has the same orthogonality property.

Exercise 2

Let P_2 denote the real vector space of polynomials of degree at most 2. For $f, g \in P_2$, define
$$\langle f, g\rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx.$$

(a) Prove that this is an inner product on P_2.

(b) Show that
$$\left\{1,\ 1-x,\ 1-2x+\frac{1}{2}x^2\right\}$$
is an orthonormal system for this inner product.

Solution

The integral is finite for all $f, g \in P_2$, because a polynomial multiplied by e^{-x} is absolutely integrable on $[0,\infty)$. Symmetry and bilinearity follow directly from the integral.

For positive definiteness,
$$\langle f, f\rangle = \int_0^\infty f(x)^2 e^{-x}\,dx \geq 0.$$
If this integral is zero, then the continuous nonnegative function $f(x)^2e^{-x}$ vanishes identically on $[0,\infty)$. Thus $f(x) = 0$ for every $x \geq 0$, and therefore the polynomial f is identically zero. Hence this is an inner product.

We use the standard moments
$$\int_0^\infty x^k e^{-x}\,dx = k!, \qquad k = 0, 1, 2, \ldots.$$
Set
$$q_0 = 1, \qquad q_1 = 1 - x, \qquad q_2 = 1 - 2x + \frac{1}{2}x^2.$$
Then
$$\langle q_0, q_0\rangle = 1, \qquad \langle q_0, q_1\rangle = 1 - 1 = 0,$$
and
$$\langle q_1, q_1\rangle = 1 - 2(1) + 2 = 1.$$

Also,

$$\langle q_0, q_2\rangle = 1 - 2(1) + \frac{1}{2}(2) = 0.$$

Next,

$$q_1 q_2 = 1 - 3x + \frac{5}{2}x^2 - \frac{1}{2}x^3,$$

so

$$\langle q_1, q_2\rangle = 1 - 3 + \frac{5}{2}(2) - \frac{1}{2}(6) = 0.$$

Finally,

$$q_2^2 = 1 - 4x + 5x^2 - 2x^3 + \frac{1}{4}x^4,$$

and therefore

$$\langle q_2, q_2\rangle = 1 - 4 + 5(2) - 2(6) + \frac{1}{4}(24) = 1.$$

Thus the three functions are pairwise orthogonal and each has norm 1. Hence

$$\boxed{\left\{1,\ 1-x,\ 1-2x+\frac{1}{2}x^2\right\}}$$

is an orthonormal system.

Exercise 3

Let $C^1[0,1]$ be the complex vector space of continuously differentiable functions on $[0,1]$.

(a) Prove that

$$\langle f, g\rangle = f(0)\overline{g(0)} + \int_0^1 f'(x)\overline{g'(x)}\,dx$$

defines an inner product on $C^1[0,1]$.

(b) Find an orthonormal system $\{h_1, h_2, h_3\}$ such that

$$\operatorname{Span}\{h_1, h_2, h_3\} = \operatorname{Span}\{1, x, x^2\}.$$

Solution

Conjugate symmetry and sesquilinearity follow immediately from the formula. Moreover,

$$\langle f, f\rangle = |f(0)|^2 + \int_0^1 |f'(x)|^2\,dx \geq 0.$$

If $\langle f, f\rangle = 0$, then $f(0) = 0$ and $f'(x) = 0$ for every $x \in [0,1]$. Hence f is constant, and because $f(0) = 0$, we obtain $f \equiv 0$. Thus the form is positive definite and is therefore an inner product.

We apply Gram–Schmidt to $1, x, x^2$.

First,

$$\|1\|^2 = |1|^2 + \int_0^1 0\,dx = 1,$$

so take

$$h_1(x) = 1.$$

Next,

$$\langle x, 1\rangle = x(0)\cdot 1 + \int_0^1 1\cdot 0\,dx = 0,$$

and

$$\|x\|^2 = 0 + \int_0^1 1\,dx = 1.$$

Thus

$$h_2(x) = x.$$

For x^2,

$$\left\langle x^2, 1 \right\rangle = 0, \qquad \left\langle x^2, x \right\rangle = \int_0^1 2x\,dx = 1.$$

Therefore the orthogonalized third vector is

$$v_3(x) = x^2 - x.$$

Its norm is

$$\|v_3\|^2 = \int_0^1 (2x-1)^2\,dx = \frac{4}{3} - 2 + 1 = \frac{1}{3}.$$

Hence

$$h_3(x) = \sqrt{3}\,(x^2 - x).$$

Consequently, one suitable orthonormal system is

$$\boxed{h_1(x) = 1, \qquad h_2(x) = x, \qquad h_3(x) = \sqrt{3}\,(x^2 - x)}.$$

The change from $\{1, x, x^2\}$ to this system is triangular with nonzero diagonal entries, so both systems span the same three-dimensional subspace.

Chapter 1, Section 4
Best Approximation and Orthogonal Projection

Projection principle

Let W be a finite-dimensional subspace of an inner-product space. For every f, there is a unique $p \in W$ minimizing $\|f-p\|$. It is characterized by

$$f - p \perp W.$$

If $e_1, \ldots, e_m$ is an orthogonal basis of W, then

$$p = \sum_{j=1}^{m} \frac{\langle f, e_j \rangle}{\|e_j\|^2} e_j.$$

Moreover, for every $w \in W$,

$$\|f-w\|^2 = \|f-p\|^2 + \|p-w\|^2,$$

so the minimizer is unique. We use this fact repeatedly below.

Exercise 1

Let $f \in C[-\pi, \pi]$. For $\alpha, \beta, \gamma \in \mathbb{C}$, define

$$F(\alpha, \beta, \gamma) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x) - \alpha - \beta \cos x - \gamma \cos 10x|^2 \, dx.$$

Prove that F has a unique minimum and find the minimizing point for each listed function.

Use the inner product

$$\langle u, v \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x)\overline{v(x)} \, dx.$$

The functions $1, \cos x, \cos 10x$ are mutually orthogonal, with

$$\|1\|^2 = 2, \qquad \|\cos x\|^2 = \|\cos 10x\|^2 = 1.$$

Therefore the unique minimizer is

$$\boxed{\alpha_0 = \frac{1}{2} \langle f, 1 \rangle, \qquad \beta_0 = \langle f, \cos x \rangle, \qquad \gamma_0 = \langle f, \cos 10x \rangle.}$$

(a) $f(x) = \cos^2 x$. Since $\cos^2 x = (1 + \cos 2x)/2$,

$$\boxed{(\alpha_0, \beta_0, \gamma_0) = \left(\frac{1}{2}, 0, 0\right).}$$

(b) $f(x) = x^3$. This function is odd, while all three basis functions are even. Hence

$$\boxed{(\alpha_0, \beta_0, \gamma_0) = (0, 0, 0).}$$

(c) $f(x) = \sin x$. Again, oddness gives

$$\boxed{(\alpha_0, \beta_0, \gamma_0) = (0, 0, 0).}$$

(d) $f(x) = 1 - 2\cos x$. It already belongs to the approximation space, so

$$\boxed{(\alpha_0, \beta_0, \gamma_0) = (1, -2, 0).}$$

(e) $f(x) = |x|$. We compute

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|\, dx = \frac{\pi}{2}.$$

Also,

$$\beta_0 = \frac{2}{\pi} \int_0^{\pi} x \cos x\, dx = \frac{2}{\pi} \left[x \sin x + \cos x\right]_0^{\pi} = -\frac{4}{\pi}.$$

For the last coefficient,

$$\gamma_0 = \frac{2}{\pi} \int_0^{\pi} x \cos 10x\, dx = \frac{2}{\pi} \left[\frac{x \sin 10x}{10} + \frac{\cos 10x}{100}\right]_0^{\pi} = 0.$$

Thus

$$\boxed{(\alpha_0, \beta_0, \gamma_0) = \left(\frac{\pi}{2}, -\frac{4}{\pi}, 0\right).}$$

(f) $f(x) = |\sin x|$. First,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin x|\, dx = \frac{2}{\pi}.$$

The coefficient of $\cos x$ vanishes by the symmetry $x \mapsto \pi - x$ on $[0, \pi]$. Finally, using

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1},$$

the coefficient of $\cos 10x$ (corresponding to $n = 5$) is

$$\gamma_0 = -\frac{4}{\pi(4 \cdot 25 - 1)} = -\frac{4}{99\pi}.$$

Therefore

$$\boxed{(\alpha_0, \beta_0, \gamma_0) = \left(\frac{2}{\pi}, 0, -\frac{4}{99\pi}\right).}$$

Exercise 2

On $C[0, 2\pi]$, use

$$\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}\, dx.$$

Consider

$$S = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(x/2)}{\sqrt{\pi}}, \frac{\sin(x/2)}{\sqrt{\pi}} \right\}.$$

(a) Orthonormality

Each function has norm one:

$$\int_0^{2\pi} \frac{dx}{2\pi} = 1, \qquad \frac{1}{\pi}\int_0^{2\pi} \cos^2(x/2)\,dx = 1, \qquad \frac{1}{\pi}\int_0^{2\pi} \sin^2(x/2)\,dx = 1.$$

The mixed inner products vanish by elementary trigonometric integration. Hence S is orthonormal.

(b) Best approximation to $f(x) = x$

Let $W = \operatorname{span} S$. Since S is orthonormal, the projection is

$$g = \sum_{e \in S} \langle x, e \rangle\, e.$$

The constant part is

$$\left\langle x, 1/\sqrt{2\pi} \right\rangle \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \pi.$$

Next,

$$\int_0^{2\pi} x\cos(x/2)\,dx = -8, \qquad \int_0^{2\pi} x\sin(x/2)\,dx = 4\pi.$$

Therefore

$$\boxed{g(x) = \pi - \frac{8}{\pi}\cos\frac{x}{2} + 4\sin\frac{x}{2}.}$$

Exercise 3

In $C[-\pi,\pi]$, let

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx, \qquad W = \operatorname{span}\{1, \sin x, \cos x, x\},$$

and let $f(x) = |x|$.

The function f is even. The odd subspace $\operatorname{span}\{\sin x, x\}$ is orthogonal to every even function, so its projection coefficients are zero. The even functions 1 and $\cos x$ are orthogonal. From Exercise 1,

$$\frac{\langle f, 1\rangle}{\|1\|^2} = \frac{\pi}{2}, \qquad \frac{\langle f, \cos x\rangle}{\|\cos x\|^2} = -\frac{4}{\pi}.$$

Thus the unique closest function is

$$\boxed{g(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x.}$$

Exercise 4

For $\alpha, \beta, \gamma \in \mathbb{C}$, define

$$F(\alpha,\beta,\gamma) = \frac{1}{\pi}\int_{-\pi}^{\pi} |x - \alpha - \beta\cos x - \gamma\sin 2x|^2\,dx.$$

The functions $1, \cos x, \sin 2x$ are mutually orthogonal, with squared norms $2, 1, 1$. Since x is odd,

$$\langle x, 1\rangle = \langle x, \cos x\rangle = 0.$$

Moreover,

$$\langle x, \sin 2x\rangle = \frac{2}{\pi}\int_0^\pi x\sin 2x\,dx = \frac{2}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^\pi = -1.$$

Hence the unique minimizer is

$$\boxed{(\alpha_0, \beta_0, \gamma_0) = (0, 0, -1).}$$

Exercise 5

In $C[-1,1]$, use

$$\langle f, g\rangle = \int_{-1}^1 f(x)\overline{g(x)}\,dx.$$

Suppose

$$\begin{aligned} f_0(x) &= 1, \\ f_1(x) &= x + a, \\ f_2(x) &= x^2 + bx + c, \\ f_3(x) &= x^3 + Ax^2 + Bx + C \end{aligned}$$

form an orthogonal system.

(a) Determine the coefficients

From $f_1 \perp f_0$,

$$0 = \int_{-1}^1 (x + a)\,dx = 2a, \qquad a = 0.$$

From $f_2 \perp f_0$ and $f_2 \perp f_1 = x$,

$$\frac{2}{3} + 2c = 0, \qquad \frac{2}{3}b = 0,$$

so

$$\boxed{a = 0, \qquad b = 0, \qquad c = -\frac{1}{3}.}$$

For completeness, orthogonality also forces

$$A = C = 0, \qquad B = -\frac{3}{5},$$

so

$$f_3(x) = x^3 - \frac{3}{5}x.$$

(b) Best approximation of x^4

Define

$$F(\alpha, \beta, \gamma, \delta) = \int_{-1}^1 \left|x^4 - \alpha f_0 - \beta f_1 - \gamma f_2 - \delta f_3\right|^2 dx.$$

Because the f_j are nonzero and orthogonal, the unique minimizing coefficients are the projection coefficients.

Since x^4 is even while f_1 and f_3 are odd,

$$\beta_0 = \delta_0 = 0.$$

For $f_0 = 1$,

$$\alpha_0 = \frac{\int_{-1}^{1} x^4 dx}{\int_{-1}^{1} 1\, dx} = \frac{2/5}{2} = \frac{1}{5}.$$

For $f_2 = x^2 - 1/3$,

$$\left\langle x^4, f_2 \right\rangle = \int_{-1}^{1} \left(x^6 - \frac{1}{3}x^4 \right) dx = \frac{2}{7} - \frac{2}{15} = \frac{16}{105},$$

and

$$\|f_2\|^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3} \right)^2 dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}.$$

Thus

$$\gamma_0 = \frac{16/105}{8/45} = \frac{6}{7}.$$

Therefore

$$\boxed{(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{1}{5}, 0, \frac{6}{7}, 0 \right).}$$

Equivalently, the best approximation is

$$\boxed{p(x) = \frac{1}{5} + \frac{6}{7}\left(x^2 - \frac{1}{3} \right) = \frac{6}{7}x^2 - \frac{3}{35}.}$$

Chapter 1, Section 5
Orthogonal Expansions and Norm Convergence

Conventions

Throughout, V is a real or complex inner-product space. In the complex case we use the convention that the inner product is linear in the first variable and conjugate-linear in the second. The induced norm is

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Exercise 1

Let $\{u_n\}_{n=1}^{\infty}$ be an orthogonal system in an inner-product space V, and suppose that

$$v = \sum_{n=1}^{\infty} a_n u_n.$$

Prove that, for every n,

$$a_n = \frac{\langle v, u_n\rangle}{\|u_n\|^2}.$$

Solution. Let

$$S_N = \sum_{k=1}^{N} a_k u_k.$$

By assumption, $S_N \to v$ in norm. Fix n. Since the map

$$x \longmapsto \langle x, u_n\rangle$$

is continuous, we obtain

$$\langle v, u_n\rangle = \lim_{N\to\infty} \langle S_N, u_n\rangle.$$

For every $N \ge n$, orthogonality gives

$$\langle S_N, u_n\rangle = \sum_{k=1}^{N} a_k \langle u_k, u_n\rangle = a_n \langle u_n, u_n\rangle = a_n \|u_n\|^2.$$

Hence

$$\langle v, u_n\rangle = a_n \|u_n\|^2.$$

Because an orthogonal system contains no zero vector, $\|u_n\|^2 > 0$, and therefore

$$\boxed{a_n = \frac{\langle v, u_n\rangle}{\|u_n\|^2}}.$$

Exercise 2

For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \dfrac{1}{n}, \\ \sqrt{n}, & \dfrac{1}{n} < x < \dfrac{2}{n}, \\ 0, & \dfrac{2}{n} \le x \le 1. \end{cases}$$

Prove that f_n converges pointwise to 0 on $[0,1]$, but does not converge to 0 in the norm

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2}.$$

Solution. Fix $x \in [0,1]$.

If $x = 0$, then $f_n(0) = 0$ for every n.

If $x > 0$, choose $N > 2/x$. Then for every $n \ge N$,

$$\frac{2}{n} < x,$$

so x lies outside the interval $(1/n, 2/n)$. Thus $f_n(x) = 0$ for all sufficiently large n. Consequently,

$$f_n(x) \longrightarrow 0 \qquad \text{for every } x \in [0,1].$$

On the other hand,

$$\|f_n\|_2^2 = \int_0^1 |f_n(x)|^2 \, dx = \int_{1/n}^{2/n} n \, dx = n \left(\frac{2}{n} - \frac{1}{n} \right) = 1.$$

Therefore

$$\|f_n\|_2 = 1 \qquad \text{for every } n.$$

Hence $\|f_n - 0\|_2$ does not tend to zero. Thus pointwise convergence does not imply convergence in the L^2 norm.

Exercise 3

Find a sequence of functions in $C[0,\infty)$ that converges uniformly to 0 on $[0,\infty)$ but does not converge to 0 in the norm

$$\|f\|_2 = \left(\int_0^\infty |f(x)|^2 \, dx \right)^{1/2}.$$

Solution. Define

$$f_n(x) = \frac{1}{\sqrt{n}} e^{-x/n}, \qquad x \ge 0.$$

Each f_n is continuous on $[0,\infty)$. Moreover,

$$\sup_{x \ge 0} |f_n(x)| = \frac{1}{\sqrt{n}},$$

because $e^{-x/n} \le 1$. Hence

$$\sup_{x\ge 0} |f_n(x)| \longrightarrow 0,$$

so $f_n \to 0$ uniformly on $[0, \infty)$.

However,

$$\|f_n\|_2^2 = \int_0^\infty \frac{1}{n} e^{-2x/n}\, dx.$$

With $u = 2x/n$, so $dx = (n/2)du$, we obtain

$$\|f_n\|_2^2 = \frac{1}{n} \cdot \frac{n}{2} \int_0^\infty e^{-u}\, du = \frac{1}{2}.$$

Therefore

$$\|f_n\|_2 = \frac{1}{\sqrt{2}} \qquad \text{for every } n.$$

Thus f_n does not converge to 0 in the L^2 norm.

Exercise 4

Let V be an inner-product space, and let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal system in V. Define

$$f_{2n-1} = \frac{e_{2n} - e_{2n-1}}{\sqrt{2}}, \qquad f_{2n} = \frac{e_{2n} + e_{2n-1}}{\sqrt{2}}, \qquad n = 1, 2, \dots$$

Prove that $\{f_n\}_{n=1}^\infty$ is also a complete orthonormal system in V.

Solution. We first prove orthonormality. For each n,

$$\|f_{2n-1}\|^2 = \frac{1}{2}\|e_{2n} - e_{2n-1}\|^2 = \frac{1}{2}(\|e_{2n}\|^2 + \|e_{2n-1}\|^2) = 1,$$

and similarly $\|f_{2n}\| = 1$.

Within the same pair,

$$\langle f_{2n-1}, f_{2n}\rangle = \frac{1}{2}\langle e_{2n} - e_{2n-1}, e_{2n} + e_{2n-1}\rangle = 0.$$

If $m \ne n$, every inner product between one of f_{2m-1}, f_{2m} and one of f_{2n-1}, f_{2n} is zero because it is a linear combination of inner products of distinct vectors from the orthonormal family $\{e_k\}$. Hence $\{f_n\}$ is orthonormal.

It remains to prove completeness. Solving the defining relations for the original basis vectors gives

$$e_{2n} = \frac{f_{2n-1} + f_{2n}}{\sqrt{2}}, \qquad e_{2n-1} = \frac{f_{2n} - f_{2n-1}}{\sqrt{2}}.$$

Thus every e_k belongs to the linear span of $\{f_n\}$. Therefore

$$\overline{\text{span}}\{e_n : n \ge 1\} \subseteq \overline{\text{span}}\{f_n : n \ge 1\}.$$

The left-hand side is V because $\{e_n\}$ is complete. Hence the closed span of $\{f_n\}$ is all of V, proving completeness.

Therefore $\{f_n\}_{n=1}^\infty$ is a complete orthonormal system.

Exercise 5

Let V be an inner-product space.

(a) Prove that for every $u, v \in V$,

$$\big|\|u\| - \|v\|\big| \le \|u - v\|.$$

(b) Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in V converging in norm to $u \in V$. Prove that

$$\lim_{n\to\infty} \|u_n\| = \|u\|.$$

Solution. **(a)** By the triangle inequality,

$$\|u\| = \|(u - v) + v\| \le \|u - v\| + \|v\|.$$

Hence

$$\|u\| - \|v\| \le \|u - v\|.$$

Interchanging u and v gives

$$\|v\| - \|u\| \le \|u - v\|.$$

Combining the two inequalities yields the reverse triangle inequality

$$\boxed{\big|\|u\| - \|v\|\big| \le \|u - v\|}.$$

(b) Since $u_n \to u$ in norm,

$$\|u_n - u\| \longrightarrow 0.$$

Applying part (a) with u_n and u, we get

$$\big|\|u_n\| - \|u\|\big| \le \|u_n - u\|.$$

The right-hand side tends to zero, so by the squeeze theorem,

$$\big|\|u_n\| - \|u\|\big| \longrightarrow 0.$$

Therefore

$$\boxed{\|u_n\| \longrightarrow \|u\|}.$$

Summary

The exercises illustrate several distinctions and structural facts that recur throughout Hilbert-space theory:

- coefficients in an orthogonal expansion are obtained by orthogonal projection;
- pointwise, uniform, and norm convergence are genuinely different notions;
- orthonormal bases remain complete under orthogonal changes of coordinates in finite blocks;
- the norm is a continuous function on every normed space.

Chapter 1 – Summary Exercises

Orthogonal Functions, Classical Polynomials, and the Discrete Fourier Basis

This document was generated by **GPT-5.6 Sol** (extended) which translated and solved the exercises in the supplied section in our **Fourier Series and Laplace Transforms** course textbook (Hebrew):
https://www.samyzaf.com/technion/fourier/fourier.pdf
AI chatbots may make errors. If you find any, please send report to mail@samyzaf.com.

Exercise 1: Monomials on the unit disk

Let

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

and let $C(D)$ be the space of continuous functions $f : D \to \mathbb{C}$. Define

$$\langle f, g\rangle = \iint_D f(x, y)\overline{g(x, y)}\, dx\, dy.$$

For each $n \geq 0$, let

$$f_n(x, y) = (x + iy)^n.$$

(a) Verification of the inner-product axioms

Conjugate symmetry follows from

$$\overline{\langle g, f\rangle} = \overline{\iint_D g\overline{f}} = \iint_D f\overline{g} = \langle f, g\rangle\,.$$

Linearity in the first argument follows from linearity of the integral. Finally,

$$\langle f, f\rangle = \iint_D |f(x, y)|^2\, dx\, dy \geq 0.$$

If this integral is zero, then the continuous nonnegative function $|f|^2$ is identically zero on D, hence $f \equiv 0$. Thus this is an inner product.

(b) Orthogonality of $\{f_n\}_{n=0}^{\infty}$

Use polar coordinates $x + iy = re^{i\theta}$. Then

$$\langle f_n, f_m\rangle = \int_0^1 \int_0^{2\pi} r^n e^{in\theta} r^m e^{-im\theta}\, r\, d\theta\, dr.$$

Therefore

$$\langle f_n, f_m\rangle = \left(\int_0^1 r^{n+m+1}\, dr\right)\left(\int_0^{2\pi} e^{i(n-m)\theta}\, d\theta\right).$$

The angular integral vanishes when $n \neq m$. Hence the family is orthogonal. For $n = m$,

$$\|f_n\|^2 = 2\pi \int_0^1 r^{2n+1}\, dr = \frac{\pi}{n+1}.$$

Thus

$$\boxed{\langle f_n, f_m\rangle = \frac{\pi}{n+1}\,\delta_{nm}.}$$

Exercise 2: Chebyshev polynomials

For $f, g \in C[-1,1]$, define

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)\overline{g(x)}}{\sqrt{1-x^2}}\, dx.$$

For $n \geq 0$, define

$$T_n(x) = \cos(n \arccos x).$$

(a) This is an inner product

The weight $(1-x^2)^{-1/2}$ is positive and integrable on $[-1,1]$. Sesquilinearity and conjugate symmetry are immediate. Moreover,

$$\langle f, f\rangle = \int_{-1}^{1} \frac{|f(x)|^2}{\sqrt{1-x^2}}\, dx \geq 0.$$

If this equals zero, continuity of f and positivity of the weight on $(-1,1)$ imply $f \equiv 0$. Hence the expression defines an inner product.

(b) Orthogonality of the Chebyshev family

Set $x = \cos\theta$, where $0 \leq \theta \leq \pi$. Since

$$dx = -\sin\theta\, d\theta, \qquad \sqrt{1-x^2} = \sin\theta,$$

we obtain

$$\langle T_n, T_m\rangle = \int_0^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta.$$

The cosine functions are orthogonal on $[0,\pi]$, so

$$\langle T_n, T_m\rangle = 0 \qquad (n \neq m).$$

More precisely,

$$\|T_0\|^2 = \pi, \qquad \|T_n\|^2 = \frac{\pi}{2} \quad (n \geq 1).$$

Thus $\{T_n\}$ is an orthogonal system.

(c) T_n is a polynomial of degree n

The angle-addition identity gives

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta).$$

With $x = \cos\theta$, this becomes

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Since

$$T_0(x) = 1, \qquad T_1(x) = x,$$

induction shows that each T_n is a polynomial. The recurrence also shows that its leading coefficient is nonzero and its degree is exactly n (for $n \geq 1$, the leading coefficient is 2^{n-1}).

Exercise 3: Legendre polynomials

The Legendre polynomials are defined by Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

(a) Orthogonality and norms

First suppose $m < n$. Since P_m has degree m, repeated integration by parts n times gives

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} P_m(x)\frac{d^n}{dx^n}(x^2-1)^n\,dx = 0.$$

All boundary terms vanish because $(x^2-1)^n$ has zeros of order n at $x = \pm 1$, and the final integral vanishes because $P_m^{(n)} \equiv 0$. Symmetry gives orthogonality also when $m > n$.

For the norm, integration by parts n times yields

$$\int_{-1}^{1} P_n(x)^2\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} P_n^{(n)}(x)(x^2-1)^n\,dx.$$

The leading coefficient of P_n is

$$\frac{(2n)!}{2^n (n!)^2},$$

so

$$P_n^{(n)}(x) = \frac{(2n)!}{2^n n!}.$$

Also, $(x^2-1)^n = (-1)^n(1-x^2)^n$. Hence

$$\int_{-1}^{1} P_n(x)^2\,dx = \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx.$$

Using the beta-integral evaluation

$$\int_{-1}^{1}(1-x^2)^n\,dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!},$$

we obtain

$$\boxed{\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n, \\ \dfrac{2}{2n+1}, & m = n. \end{cases}}$$

(b) Legendre expansion of a step function

Let

$$f(x) = \begin{cases} 0, & -1 \le x < 0, \\ 1, & 0 < x \le 1. \end{cases}$$

Its Legendre coefficients are

$$c_n = \frac{\langle f, P_n\rangle}{\|P_n\|^2} = \frac{2n+1}{2}\int_0^1 P_n(x)\,dx.$$

For $n = 0$,

$$c_0 = \frac{1}{2}.$$

For $n \geq 1$, use
$$(2n+1)P_n(x) = \frac{d}{dx}\big(P_{n+1}(x) - P_{n-1}(x)\big).$$
Since $P_k(1) = 1$,
$$\int_0^1 P_n(x)\,dx = \frac{P_{n-1}(0) - P_{n+1}(0)}{2n+1}.$$
Thus
$$c_n = \frac{1}{2}\big(P_{n-1}(0) - P_{n+1}(0)\big).$$
For even $n \geq 2$, both indices are odd and $P_{2j+1}(0) = 0$, so $c_n = 0$. For $n = 2k+1$,
$$P_{2k}(0) = (-1)^k \frac{\binom{2k}{k}}{4^k},$$
and therefore
$$c_{2k+1} = \frac{4k+3}{4k+4}(-1)^k \frac{\binom{2k}{k}}{4^k}.$$
Consequently,
$$\boxed{f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4k+3}{4k+4}(-1)^k \frac{\binom{2k}{k}}{4^k} P_{2k+1}(x).}$$
At points of continuity the series converges to $f(x)$; at $x = 0$ it converges to the midpoint value $1/2$.

Exercise 4: The discrete Fourier basis

Fix $N \in \mathbb{N}$. For $m = 0, 1, \ldots, N-1$, define
$$u_m = \Big(1, e^{-2\pi i m/N}, e^{-4\pi i m/N}, \ldots, e^{-2\pi i (N-1)m/N}\Big) \in \mathbb{C}^N.$$
Equip $\mathbb{C}^N$ with the standard inner product
$$\langle u, v\rangle = \sum_{k=0}^{N-1} u_k \overline{v_k}.$$

(a) Orthogonality

For $m, \ell \in \{0, \ldots, N-1\}$,
$$\langle u_m, u_\ell\rangle = \sum_{k=0}^{N-1} e^{-2\pi i k m/N} e^{2\pi i k \ell/N} = \sum_{k=0}^{N-1} q^k,$$
where
$$q = e^{2\pi i(\ell - m)/N}.$$
If $m \neq \ell$, then $q \neq 1$ but $q^N = 1$, and hence
$$\sum_{k=0}^{N-1} q^k = \frac{1 - q^N}{1 - q} = 0.$$
Thus the vectors are pairwise orthogonal.

(b) Their norms

For every m,

$$\|u_m\|^2 = \sum_{k=0}^{N-1} \left| e^{-2\pi i k m/N} \right|^2 = \sum_{k=0}^{N-1} 1 = N.$$

Hence

$$\boxed{\|u_m\|^2 = N.}$$

(c) Fourier reconstruction formula

There are N nonzero orthogonal vectors $u_0, \dots, u_{N-1}$ in the N-dimensional space $\mathbb{C}^N$, so they form an orthogonal basis. Therefore every $u \in \mathbb{C}^N$ has the expansion

$$u = \sum_{m=0}^{N-1} \frac{\langle u, u_m \rangle}{\|u_m\|^2} u_m.$$

Using $\|u_m\|^2 = N$ gives

$$\boxed{u = \frac{1}{N} \sum_{m=0}^{N-1} \langle u, u_m \rangle \; u_m.}$$

The coefficient vector

$$(\langle u, u_0 \rangle, \langle u, u_1 \rangle, \dots, \langle u, u_{N-1} \rangle)$$

is the discrete Fourier transform of u, up to the normalization convention.

Editorial note

The original sheet indexes the Chebyshev and Legendre systems beginning at $n = 1$ in some displayed notation, although T_0 and P_0 are naturally part of the standard orthogonal families. The solutions above include the zeroth polynomial whenever appropriate.

Chapter 2, Section 1 - Fourier Series

Throughout, for a function on $[-\pi, \pi]$ we use

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx.$$

Exercise 1

Find the Fourier series on $[-\pi, \pi]$ of each of the following functions.

1(a) $f(x) = |\sin x|$

The function is even, hence $b_n = 0$. Also

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x\,dx = \frac{4}{\pi}.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx\,dx.$$

The integral vanishes for odd n. For $n = 2k$,

$$\int_0^{\pi} \sin x \cos(2kx)\,dx = \frac{1}{2} \int_0^{\pi} [\sin((2k+1)x) - \sin((2k-1)x)]\,dx = -\frac{2}{4k^2-1}.$$

Thus

$$a_{2k} = -\frac{4}{\pi(4k^2-1)}, \qquad a_{2k-1} = 0,$$

and therefore

$$\boxed{|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}.}$$

The series converges to $|\sin x|$ at every real x.

1(b)

Let

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ e^x, & 0 < x \leq \pi. \end{cases}$$

Then
$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^x\,dx = \frac{e^{\pi}-1}{\pi}.$$
Using
$$\int e^x \cos nx\,dx = \frac{e^x(\cos nx + n\sin nx)}{1+n^2},$$
we obtain
$$a_n = \frac{(-1)^n e^{\pi} - 1}{\pi(1+n^2)}.$$
Similarly,
$$\int e^x \sin nx\,dx = \frac{e^x(\sin nx - n\cos nx)}{1+n^2},$$
so
$$b_n = \frac{n(1-(-1)^n e^{\pi})}{\pi(1+n^2)}.$$
Hence
$$\boxed{f(x) \sim \frac{e^{\pi}-1}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n e^{\pi}-1}{\pi(1+n^2)}\cos nx + \frac{n(1-(-1)^n e^{\pi})}{\pi(1+n^2)}\sin nx\right].}$$
At $x = 0$ the periodic extension has a jump from 0 to 1, so the series converges there to $1/2$. At $x = \pm\pi$ it converges to $(e^{\pi}+0)/2 = e^{\pi}/2$.

1(c)

Let
$$f(x) = \begin{cases} \cos x, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases}$$
First,
$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{0}\cos x\,dx + \int_0^{\pi}\sin x\,dx\right) = \frac{2}{\pi}.$$
For the cosine coefficients,
$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}\cos x\cos nx\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right).$$
The first integral is $\pi/2$ when $n = 1$ and 0 otherwise. The second is 0 for odd n, while for even n it equals $2/(1-n^2)$. Consequently,
$$a_1 = \frac{1}{2}, \qquad a_{2k} = \frac{2}{\pi(1-4k^2)}, \qquad a_{2k+1} = 0 \quad (k \geq 1).$$
For the sine coefficients,
$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}\cos x\sin nx\,dx + \int_0^{\pi}\sin x\sin nx\,dx\right).$$
The second integral is $\pi/2$ for $n = 1$ and 0 otherwise. For even n the first integral equals $-2n/(n^2-1)$, and for odd n it is 0. Thus
$$b_1 = \frac{1}{2}, \qquad b_{2k} = -\frac{4k}{\pi(4k^2-1)}, \qquad b_{2k+1} = 0 \quad (k \geq 1).$$
Therefore
$$\boxed{\begin{aligned} f(x) \sim{}& \frac{1}{\pi} + \frac{1}{2}(\cos x + \sin x) \\ &- \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{k\sin(2kx)}{4k^2-1}. \end{aligned}}$$

1(d)

Let
$$f(x) = \begin{cases} 0, & -\pi \le x \le 0, \\ x, & 0 < x \le \pi. \end{cases}$$
Then
$$a_0 = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}.$$
Integration by parts gives
$$a_n = \frac{1}{\pi}\int_0^\pi x\cos nx\,dx = \frac{(-1)^n - 1}{\pi n^2},$$
and
$$b_n = \frac{1}{\pi}\int_0^\pi x\sin nx\,dx = \frac{(-1)^{n+1}}{n}.$$
Hence
$$\boxed{f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right].}$$

Exercise 2

For a fixed real number p, $-\pi \le p \le \pi$, define
$$f_p(x) = \begin{cases} 0, & -\pi \le x \le p, \\ 1, & p < x \le \pi. \end{cases}$$
Find its Fourier series.

We have
$$a_0 = \frac{1}{\pi}\int_p^\pi 1\,dx = \frac{\pi - p}{\pi}.$$
For $n \ge 1$,
$$a_n = \frac{1}{\pi}\int_p^\pi \cos nx\,dx = -\frac{\sin(np)}{\pi n},$$
and
$$b_n = \frac{1}{\pi}\int_p^\pi \sin nx\,dx = \frac{\cos(np) - (-1)^n}{\pi n}.$$
Therefore
$$\boxed{f_p(x) \sim \frac{\pi - p}{2\pi} + \sum_{n=1}^{\infty}\left[-\frac{\sin(np)}{\pi n}\cos nx + \frac{\cos(np) - (-1)^n}{\pi n}\sin nx\right].}$$
At $x = p$ the series converges to $1/2$. At the endpoints it converges to the average of the one-sided values of the periodic extension.

Exercise 3

Let f be piecewise continuous and 2π-periodic, with Fourier series
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

3(a): Translation by π

Define $g(x) = f(x + \pi)$ and write

$$g(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx).$$

Since

$$\cos(n(x + \pi)) = (-1)^n \cos nx, \qquad \sin(n(x + \pi)) = (-1)^n \sin nx,$$

we immediately obtain

$$\boxed{A_0 = a_0, \qquad A_n = (-1)^n a_n, \qquad B_n = (-1)^n b_n.}$$

3(b): Multiplication by $\cos x$

Define $h(x) = f(x) \cos x$ and write

$$h(x) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx).$$

Use the product identities

$$\cos kx \cos x = \frac{1}{2}(\cos((k + 1)x) + \cos((k - 1)x)),$$

$$\sin kx \cos x = \frac{1}{2}(\sin((k + 1)x) + \sin((k - 1)x)).$$

The constant term comes from $a_1 \cos^2 x$, so

$$\boxed{\alpha_0 = a_1.}$$

The coefficient of $\cos x$ is

$$\boxed{\alpha_1 = \frac{a_0 + a_2}{2},}$$

while for $n \geq 2$,

$$\boxed{\alpha_n = \frac{a_{n-1} + a_{n+1}}{2}.}$$

For the sine coefficients,

$$\boxed{\beta_1 = \frac{b_2}{2}, \qquad \beta_n = \frac{b_{n-1} + b_{n+1}}{2} \quad (n \geq 2).}$$

Equivalently, the latter formula is valid for all $n \geq 1$ if one defines $b_0 = 0$.

Exercise 4

Let θ be fixed. Prove that

$$\left\{\frac{1}{\sqrt{2}}, \cos(x + \theta), \sin(x + \theta), \cos(2x + \theta), \sin(2x + \theta), \ldots\right\}$$

is an orthonormal system in E with inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\, dx.$$

First,

$$\left\|\frac{1}{\sqrt{2}}\right\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\,dx = 1.$$

For every $n \geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(nx+\theta)\,dx = 1, \qquad \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(nx+\theta)\,dx = 1.$$

The constant function is orthogonal to every sine and cosine because each such function has integral zero over an interval of length 2π.

For $m \neq n$, product-to-sum identities show that every product of two functions with frequencies m and n is a sum of sines or cosines of nonzero integer frequencies, hence has integral zero. For equal frequencies,

$$\int_{-\pi}^{\pi}\cos(nx+\theta)\sin(nx+\theta)\,dx = 0.$$

Thus all distinct members are orthogonal and every member has norm one. Therefore the displayed family is orthonormal.

Exercise 5

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

Show that the same series can be written in phase-amplitude form.

Cosine phase form

We seek real numbers A_n and phases $-\pi/2 < \alpha_n \leq \pi/2$ such that

$$a_n\cos nx + b_n\sin nx = A_n\cos(nx-\alpha_n).$$

Because

$$A_n\cos(nx-\alpha_n) = A_n\cos\alpha_n\cos nx + A_n\sin\alpha_n\sin nx,$$

we need

$$a_n = A_n\cos\alpha_n, \qquad b_n = A_n\sin\alpha_n.$$

The restriction on α_n means that A_n is allowed to be signed. Define:

$$\begin{cases} \alpha_n = \arctan(b_n/a_n), \quad A_n = a_n/\cos\alpha_n, & a_n \neq 0, \\ \alpha_n = \pi/2, \quad A_n = b_n, & a_n = 0,\ b_n \neq 0, \\ \alpha_n = 0, \quad A_n = 0, & a_n = b_n = 0. \end{cases}$$

Also set $A_0 = a_0/2$. Then

$$\boxed{f(x) \sim A_0 + \sum_{n=1}^{\infty}A_n\cos(nx-\alpha_n), \qquad -\frac{\pi}{2} < \alpha_n \leq \frac{\pi}{2}.}$$

Sine phase form

Similarly, seek

$$a_n \cos nx + b_n \sin nx = B_n \sin(nx + \beta_n).$$

Since

$$B_n \sin(nx + \beta_n) = B_n \sin \beta_n \cos nx + B_n \cos \beta_n \sin nx,$$

we require

$$a_n = B_n \sin \beta_n, \qquad b_n = B_n \cos \beta_n.$$

Choose

$$\begin{cases} \beta_n = \arctan(a_n/b_n), \quad B_n = b_n/\cos \beta_n, & b_n \neq 0, \\ \beta_n = \pi/2, \quad B_n = a_n, & b_n = 0,\ a_n \neq 0, \\ \beta_n = 0, \quad B_n = 0, & a_n = b_n = 0. \end{cases}$$

Set $B_0 = a_0/2$. Then

$$\boxed{f(x) \sim B_0 + \sum_{n=1}^{\infty} B_n \sin(nx + \beta_n), \qquad -\frac{\pi}{2} < \beta_n \leq \frac{\pi}{2}.}$$

Editorial note

In Exercise 5, the phase interval $(-\pi/2, \pi/2]$ cannot in general be maintained simultaneously with the conventional requirement $A_n, B_n \geq 0$. The formulation above follows the printed phase restriction and therefore permits signed amplitudes. If nonnegative amplitudes are preferred, one instead takes

$$A_n = B_n = \sqrt{a_n^2 + b_n^2}$$

and allows phases in a full interval of length 2π.

Chapter 2, Section 2 – Fourier Series

Throughout, Fourier series on $[-\pi, \pi]$ are written as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

Exercise 1

Find the Fourier series of each of the following functions on $[-\pi, \pi]$:

(a) $f(x) = x|x|$;

(b) $f(x) = \pi^2 - x^2$;

(c) $f(x) = \sin(x/2)$.

Solution. **(a)** Since $f(x) = x|x|$ is odd, $a_0 = a_n = 0$. Thus only the sine coefficients occur:

$$b_n = \frac{2}{\pi} \int_0^{\pi} x^2 \sin(nx)\, dx.$$

Twice integrating by parts gives

$$\int_0^{\pi} x^2 \sin(nx)\, dx = -\frac{\pi^2(-1)^n}{n} + \frac{2((-1)^n - 1)}{n^3}.$$

Therefore

$$\boxed{b_n = -\frac{2\pi(-1)^n}{n} + \frac{4((-1)^n - 1)}{\pi n^3}}$$

and hence

$$\boxed{x|x| \sim \sum_{n=1}^{\infty} \left[-\frac{2\pi(-1)^n}{n} + \frac{4((-1)^n - 1)}{\pi n^3} \right] \sin(nx).}$$

(b) The function $f(x) = \pi^2 - x^2$ is even, so $b_n = 0$. First,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi^2 - x^2)\, dx = \frac{4\pi^2}{3}.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi^2 - x^2) \cos(nx)\, dx.$$

Since $\int_0^\pi \cos(nx)\,dx = 0$ and

$$\int_0^\pi x^2\cos(nx)\,dx = \frac{2\pi(-1)^n}{n^2},$$

we obtain

$$a_n = -\frac{4(-1)^n}{n^2}.$$

Thus

$$\boxed{\pi^2 - x^2 \sim \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(nx).}$$

(c) The function $f(x) = \sin(x/2)$ is odd, hence $a_0 = a_n = 0$. For $n \geq 1$,

$$b_n = \frac{2}{\pi}\int_0^\pi \sin\frac{x}{2}\sin(nx)\,dx.$$

Using the product-to-sum identity,

$$\int_0^\pi \sin\frac{x}{2}\sin(nx)\,dx = \frac{1}{2}\left[\frac{\sin((n-\frac{1}{2})\pi)}{n-\frac{1}{2}} - \frac{\sin((n+\frac{1}{2})\pi)}{n+\frac{1}{2}}\right].$$

Since

$$\sin\left(n-\frac{1}{2}\right)\pi = (-1)^{n+1}, \qquad \sin\left(n+\frac{1}{2}\right)\pi = (-1)^n,$$

it follows that

$$b_n = \frac{8(-1)^{n+1}n}{\pi(4n^2-1)}.$$

Therefore

$$\boxed{\sin\frac{x}{2} \sim \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{4n^2-1}\sin(nx).}$$

At the endpoints, the periodic extension has a jump, so the series converges there to the midpoint value 0.

Exercise 2

For each real number p with $0 \leq p \leq \pi$, find the Fourier series of

$$f_p(x) = \cos(px)$$

on $[-\pi, \pi]$.

Solution. The function is even, so $b_n = 0$. For $p \neq 0$,

$$a_0 = \frac{2}{\pi}\int_0^\pi \cos(px)\,dx = \frac{2\sin(\pi p)}{\pi p}.$$

Thus the constant term is

$$\frac{a_0}{2} = \frac{\sin(\pi p)}{\pi p}.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \cos(px)\cos(nx)\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin((p-n)\pi)}{p-n} + \frac{\sin((p+n)\pi)}{p+n}\right]. \end{aligned}$$

If $p \notin \mathbb{Z}$, then

$$\sin((p \pm n)\pi) = (-1)^n \sin(\pi p),$$

and consequently

$$\boxed{a_n = \frac{2p(-1)^n \sin(\pi p)}{\pi(p^2 - n^2)}.}$$

Hence, for noninteger p,

$$\boxed{\cos(px) \sim \frac{\sin(\pi p)}{\pi p} + \frac{2p\sin(\pi p)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{p^2 - n^2} \cos(nx).}$$

The exceptional integer values are interpreted either directly or by taking limits:

$$\boxed{\begin{cases} p = 0: & f_p(x) = 1, \\ p = k \in \mathbb{N}: & f_p(x) = \cos(kx), \end{cases}}$$

so for $p = k \geq 1$ the only nonzero Fourier coefficient is $a_k = 1$.

Exercise 3

Let $f \in E$ have Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

on $[-\pi, \pi]$. Define

$$g(x) = \frac{f(x) + f(-x)}{2}, \qquad h(x) = \frac{f(x) - f(-x)}{2}.$$

Find the Fourier series of g and h.

Solution. The function g is the even part of f, while h is the odd part. Formally replacing x by $-x$ gives

$$f(-x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx - b_n \sin nx),$$

because cosine is even and sine is odd.

Adding the two series and dividing by 2 yields

$$\boxed{g(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).}$$

Subtracting and dividing by 2 yields

$$\boxed{h(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx).}$$

Thus the even part retains exactly the cosine coefficients (including the constant term), and the odd part retains exactly the sine coefficients.

Final summary

$$\begin{aligned}
x|x| &\sim \sum_{n=1}^{\infty} \left[-\frac{2\pi(-1)^n}{n} + \frac{4\big((-1)^n - 1\big)}{\pi n^3} \right] \sin(nx), \\
\pi^2 - x^2 &\sim \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(nx), \\
\sin\frac{x}{2} &\sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{4n^2 - 1} \sin(nx).
\end{aligned}$$

Chapter 2, Section 3 – Complex Fourier Series

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Throughout, the complex Fourier series of a 2π-periodic function is written as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

All coefficient identities below follow from this normalization.

Exercise 1

Let $f \in E$. Find the complex Fourier series of $\operatorname{Re} f$ by using the complex Fourier series of f.

Solution. Suppose

$$f(x) \sim \sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

Since

$$\operatorname{Re} f(x) = \frac{f(x) + \overline{f(x)}}{2},$$

we first determine the coefficients of $\overline{f(x)}$. Formally,

$$\overline{f(x)} \sim \sum_{n\in\mathbb{Z}} \overline{c_n} e^{-inx} = \sum_{n\in\mathbb{Z}} \overline{c_{-n}} e^{inx}.$$

Thus the nth complex Fourier coefficient of $\operatorname{Re} f$ is

$$\boxed{\frac{c_n + \overline{c_{-n}}}{2}}.$$

Consequently,

$$\boxed{\operatorname{Re} f(x) \sim \sum_{n=-\infty}^{\infty} \frac{c_n + \overline{c_{-n}}}{2}\, e^{inx}.}$$

Exercise 2

Let

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ e^{ix}, & 0 \le x < \pi. \end{cases}$$

Find the complex Fourier series of f.

Solution. For $n \in \mathbb{Z}$,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{0}^{\pi} e^{i(1-n)x}\,dx.$$

If $n = 1$, then
$$c_1 = \frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{1}{2}.$$
If $n \neq 1$, then
$$c_n = \frac{e^{i(1-n)\pi} - 1}{2\pi i(1-n)} = \frac{(-1)^{1-n} - 1}{2\pi i(1-n)}.$$
When n is odd and $n \neq 1$, the numerator vanishes, so $c_n = 0$. When n is even, the numerator equals -2, and therefore
$$c_n = \frac{i}{\pi(1-n)}.$$
Hence
$$\boxed{c_n = \begin{cases} \dfrac{1}{2}, & n = 1, \\ \dfrac{i}{\pi(1-n)}, & n \text{ even}, \\ 0, & n \text{ odd and } n \neq 1. \end{cases}}$$
Equivalently,
$$\boxed{f(x) \sim \frac{1}{2}e^{ix} + \sum_{k=-\infty}^{\infty} \frac{i}{\pi(1-2k)} e^{i2kx}.}$$
At the jump points, the Fourier series converges to the midpoint of the one-sided limits.

Exercise 3

Let $f \in E$, and suppose
$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$
Find the complex Fourier series of $\overline{f(x)}$, $f(-x)$, and $\overline{f(-x)}$.

Solution. For the conjugate function,
$$\overline{f(x)} \sim \sum_{n\in\mathbb{Z}} \overline{c_n} e^{-inx} = \sum_{n\in\mathbb{Z}} \overline{c_{-n}} e^{inx}.$$
Thus
$$\boxed{\overline{f(x)} \sim \sum_{n\in\mathbb{Z}} \overline{c_{-n}} e^{inx}.}$$
Replacing x by $-x$ gives
$$f(-x) \sim \sum_{n\in\mathbb{Z}} c_n e^{-inx} = \sum_{n\in\mathbb{Z}} c_{-n} e^{inx},$$
so
$$\boxed{f(-x) \sim \sum_{n\in\mathbb{Z}} c_{-n} e^{inx}.}$$
Finally,
$$\overline{f(-x)} \sim \sum_{n\in\mathbb{Z}} \overline{c_n} e^{inx},$$
and hence
$$\boxed{\overline{f(-x)} \sim \sum_{n\in\mathbb{Z}} \overline{c_n} e^{inx}.}$$

Exercise 4

Let $f, g \in E$ be 2π-periodic, with complex Fourier series

$$f(x) \sim \sum_{n\in\mathbb{Z}} a_n e^{inx}, \qquad g(x) \sim \sum_{n\in\mathbb{Z}} b_n e^{inx}.$$

For real x, define the periodic convolution

$$h(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)\,dt.$$

(a) Prove that h is piecewise continuous and 2π-periodic.

(b) If

$$h(x) \sim \sum_{n\in\mathbb{Z}} c_n e^{inx},$$

prove that $c_n = a_n b_n$ for every $n \in \mathbb{Z}$.

Solution. **(a)** Periodicity follows immediately:

$$h(x+2\pi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+2\pi-t)g(t)\,dt = h(x).$$

Because translations of a piecewise continuous function are piecewise continuous, the integrand is piecewise continuous in t for each fixed x. Standard properties of convolution on the circle imply that h is piecewise continuous; in fact, under the usual integrability assumptions it is continuous.

(b) The nth coefficient of h is

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x)e^{-inx}\,dx.$$

Substituting the definition of h and applying Fubini's theorem,

$$c_n = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x-t)g(t)e^{-inx}\,dt\,dx.$$

For fixed t, set $u = x - t$. Then

$$e^{-inx} = e^{-in(u+t)} = e^{-inu}e^{-int}.$$

Because the integrand is 2π-periodic in u, integration over any interval of length 2π gives the same value. Therefore

$$\int_{-\pi}^{\pi} f(x-t)e^{-inx}\,dx = e^{-int} \int_{-\pi}^{\pi} f(u)e^{-inu}\,du = 2\pi a_n e^{-int}.$$

Hence

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} a_n g(t)e^{-int}\,dt = a_n b_n.$$

Thus

$$\boxed{c_n = a_n b_n \qquad (n \in \mathbb{Z}).}$$

This is the convolution theorem for complex Fourier series.

Exercise 5

Find the complex Fourier series of

$$f(x) = e^x$$

on $[-\pi, \pi]$, interpreted through its 2π-periodic extension.

Solution. For $n \in \mathbb{Z}$,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)x}\, dx.$$

Thus

$$c_n = \frac{e^{(1-in)\pi} - e^{-(1-in)\pi}}{2\pi(1-in)}.$$

Since $e^{\pm in\pi} = (-1)^n$,

$$e^{(1-in)\pi} - e^{-(1-in)\pi} = (-1)^n(e^{\pi} - e^{-\pi}) = 2(-1)^n \sinh \pi.$$

Therefore

$$\boxed{c_n = \frac{(-1)^n \sinh \pi}{\pi(1-in)} = \frac{(-1)^n(1+in)\sinh \pi}{\pi(1+n^2)}.}$$

Hence

$$\boxed{e^x \sim \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh \pi}{\pi(1-in)} e^{inx}.}$$

For $-\pi < x < \pi$, the series converges to e^x. At $x = \pm\pi$, it converges to

$$\frac{e^{\pi} + e^{-\pi}}{2} = \cosh \pi.$$

Exercise 6

Suppose

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Prove the following statements.

(a) If f is real-valued, then $c_{-n} = \overline{c_n}$.

(b) If f is purely imaginary-valued, then $c_{-n} = -\overline{c_n}$.

(c) If f is real-valued and even, then every c_n is real.

(d) If f is real-valued and odd, then every c_n is purely imaginary.

Solution. **(a)** If f is real-valued, then $\overline{f(x)} = f(x)$. Therefore

$$\overline{c_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx}\, dx = c_{-n}.$$

Hence

$$\boxed{c_{-n} = \overline{c_n}.}$$

(b) If f is purely imaginary, then $\overline{f(x)} = -f(x)$. Consequently,

$$\overline{c_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)} e^{inx}\,dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx}\,dx = -c_{-n}.$$

Thus

$$\boxed{c_{-n} = -\overline{c_n}.}$$

(c) If f is even, then by substituting $x \mapsto -x$,

$$c_{-n} = c_n.$$

If f is also real-valued, part (a) gives

$$c_n = c_{-n} = \overline{c_n},$$

so c_n is real.

(d) If f is odd, then

$$c_{-n} = -c_n.$$

If f is also real-valued, part (a) gives

$$\overline{c_n} = c_{-n} = -c_n.$$

Therefore $\operatorname{Re} c_n = 0$, so c_n is purely imaginary.

Editorial Note

The typography of Exercise 3 in the source is visually compressed. The three standard transformations intended there are interpreted as

$$\overline{f(x)}, \qquad f(-x), \qquad \overline{f(-x)}.$$

These are the natural companion identities for complex Fourier coefficients and are consistent with Exercise 6.

Chapter 2, Section 4
Fourier Series: Exercises and Solutions

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Conventions

For a real Fourier series on $[-\pi, \pi]$ we use

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

At a jump discontinuity, the Fourier series converges to the midpoint of the one-sided limits of the 2π-periodic extension.

Exercise 1

Let $f(x) = 1 - x^2$ on $[-\pi, \pi]$.

(a) Find its Fourier coefficients.

(b) Determine the values to which the Fourier series converges at $x = \pi/5$ and $x = \pi/6$.

Solution. The function is even, hence $b_n = 0$. Also

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (1 - x^2)\, dx = 2 - \frac{2\pi^2}{3}.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^{\pi} (1 - x^2) \cos(nx)\, dx.$$

The integral of $\cos(nx)$ is zero, and two integrations by parts give

$$\int_0^{\pi} x^2 \cos(nx)\, dx = \frac{2\pi(-1)^n}{n^2}.$$

Therefore

$$a_n = -\frac{4(-1)^n}{n^2} = \frac{4(-1)^{n+1}}{n^2}.$$

Thus

$$\boxed{1 - x^2 \sim 1 - \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(nx).}$$

Both $\pi/5$ and $\pi/6$ are interior continuity points, so the series converges there to f itself:

$$\boxed{S(\pi/5) = 1 - \frac{\pi^2}{25}, \qquad S(\pi/6) = 1 - \frac{\pi^2}{36}.}$$

Exercise 2

For a real number $p \neq 0$, let $f_p(x) = e^{px}$ on $[-\pi, \pi]$.

(a) Find the Fourier coefficients.

(b) Compute $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} (-1)^n a_n$.

Solution. First,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{px}\, dx = \frac{2\sinh(\pi p)}{\pi p}.$$

For $n \geq 1$,

$$\int e^{px} \cos(nx)\, dx = \frac{e^{px}(p\cos nx + n \sin nx)}{p^2 + n^2},$$

so

$$\boxed{a_n = \frac{2p(-1)^n \sinh(\pi p)}{\pi(p^2 + n^2)}.}$$

Similarly,

$$\int e^{px} \sin(nx)\, dx = \frac{e^{px}(p\sin nx - n \cos nx)}{p^2 + n^2},$$

and hence

$$\boxed{b_n = \frac{2n(-1)^{n+1} \sinh(\pi p)}{\pi(p^2 + n^2)}.}$$

At $x = 0$, the function is continuous, so

$$1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.$$

Thus, if the sum starts at $n = 0$ and includes the full coefficient a_0,

$$\boxed{\sum_{n=0}^{\infty} a_n = 1 + \frac{a_0}{2} = 1 + \frac{\sinh(\pi p)}{\pi p}.}$$

At $x = \pi$, the periodic extension has one-sided limits $e^{\pi p}$ and $e^{-\pi p}$, so the Fourier series converges to

$$\frac{e^{\pi p} + e^{-\pi p}}{2} = \cosh(\pi p).$$

Therefore

$$\cosh(\pi p) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (-1)^n a_n,$$

and hence

$$\boxed{\sum_{n=0}^{\infty} (-1)^n a_n = \cosh(\pi p) + \frac{\sinh(\pi p)}{\pi p}.}$$

Exercise 3

Let f satisfy the hypotheses of Dirichlet's theorem on $[-\pi,\pi]$. Compute:

$$\text{(a)}\quad \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt,$$

$$\text{(b)}\quad \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\left(n-\frac{1}{2}\right)t\,dt,$$

$$\text{(c)}\quad \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{t}\sin\left(n-\frac{1}{2}\right)t\,dt.$$

Solution. For (a), the integral is the nth sine coefficient b_n. By the Riemann–Lebesgue lemma,

$$\boxed{\lim_{n\to\infty} b_n = 0.}$$

The same lemma applies to the frequencies $n-\frac{1}{2}\to\infty$, giving

$$\boxed{\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\left(n-\frac{1}{2}\right)t\,dt = 0.}$$

For (c), the kernel

$$K_n(t) = \frac{1}{\pi}\frac{\sin((n-\frac{1}{2})t)}{t}$$

is a Dirichlet-type approximate identity centered at 0. Under the Dirichlet hypotheses,

$$\boxed{\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{t}\sin\left(n-\frac{1}{2}\right)t\,dt = \frac{f(0+)+f(0-)}{2}.}$$

If f is continuous at 0, this reduces to $f(0)$.

Exercise 4

Define

$$f(x) = \begin{cases} x-\lfloor x\rfloor, & x\notin\mathbb{Z},\\ \frac{1}{2}, & x\in\mathbb{Z}.\end{cases}$$

Find its Fourier series on $[-\pi,\pi]$, and determine the convergence values at $x=1.5$, $x=3$, and $x=5$.

Solution. Away from integers, $f(-x)=1-f(x)$, so $f(x)-\frac{1}{2}$ is odd. Consequently,

$$a_0 = 1, \qquad a_n = 0 \quad (n\ge 1).$$

To compute b_n, it is convenient to use the jumps of the 2π-periodic extension. On each continuity interval, $f'(x)=1$. Inside $[-\pi,\pi]$ there are jumps of size -1 at the integers

$$-3,-2,-1,0,1,2,3.$$

At the endpoint of the periodic interval the jump is

$$f(-\pi+)-f(\pi-) = (4-\pi)-(\pi-3) = 7-2\pi.$$

For $n \neq 0$, the distributional derivative relation gives

$$inc_n = \frac{1}{2\pi}\left[-\sum_{k=-3}^{3} e^{-ink} + (7-2\pi)(-1)^n\right].$$

Since

$$\sum_{k=-3}^{3} e^{-ink} = 1 + 2\cos n + 2\cos 2n + 2\cos 3n,$$

we obtain $c_n = -ib_n/2$ and therefore

$$\boxed{b_n = \frac{(7-2\pi)(-1)^n - (1 + 2\cos n + 2\cos 2n + 2\cos 3n)}{\pi n}.}$$

Hence

$$\boxed{f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin(nx).}$$

At $x = 1.5$, the function is continuous and equals $1/2$, so the series converges to $1/2$. At the integer $x = 3$, the one-sided limits are 1 and 0, so the midpoint is again $1/2$. For $x = 5$, reduce modulo 2π:

$$5 - 2\pi \in (-2, -1),$$

and therefore

$$f(5-2\pi) = (5-2\pi) - (-2) = 7 - 2\pi.$$

Thus

$$\boxed{S(1.5) = \frac{1}{2}, \qquad S(3) = \frac{1}{2}, \qquad S(5) = 7 - 2\pi.}$$

Exercise 5

For $m \in \mathbb{N}$ define

$$D_m(t) = \frac{1}{2} + \sum_{n=1}^{m} \cos(nt).$$

(a) Compute $\int_{-\pi}^{\pi} D_m(t)\sin(100t)\,dt$.

(b) For $m = 100$, compute

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [D_m(t)]^2\,dt.$$

(c) Let

$$g(t) = \begin{cases} \dfrac{\sin(t/2)}{t}, & t \neq 0, \\ \dfrac{1}{2}, & t = 0. \end{cases}$$

Compute

$$\lim_{m\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} D_m(t)g(t)\,dt.$$

Solution. The function D_m is even, whereas $\sin(100t)$ is odd. Hence

$$\boxed{\int_{-\pi}^{\pi} D_m(t)\sin(100t)\,dt = 0.}$$

By orthogonality,

$$\int_{-\pi}^{\pi} D_m(t)^2\,dt = \int_{-\pi}^{\pi} \frac{1}{4}\,dt + \sum_{n=1}^{m}\int_{-\pi}^{\pi}\cos^2(nt)\,dt = \frac{\pi}{2} + m\pi.$$

Thus for $m = 100$,

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} D_{100}(t)^2\,dt = 100 + \frac{1}{2} = \frac{201}{2}.}$$

Finally,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_m(t)g(t)\,dt$$

is the mth Fourier partial sum of the even function g at $x = 0$. Since g is continuous at 0 and $g(0) = 1/2$, Dirichlet's theorem yields

$$\boxed{\lim_{m\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} D_m(t)g(t)\,dt = \frac{1}{2}.}$$

Exercise 6

Let $f(x) = xe^{ix}$ on $[-\pi, \pi]$. Find its complex Fourier series

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

and compute $F(\pi)$, $F(-\pi)$, and $F(\pi/4)$.

Solution. The complex coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{i(1-n)x}\,dx.$$

If $n = 1$, then $c_1 = 0$. If $n \neq 1$, put $k = 1 - n \neq 0$. Since

$$\int_{-\pi}^{\pi} xe^{ikx}\,dx = -\frac{2\pi i(-1)^k}{k},$$

we get

$$\boxed{c_n = -\frac{i(-1)^n}{n-1}, \qquad n \neq 1,} \qquad \boxed{c_1 = 0.}$$

Thus

$$\boxed{xe^{ix} \sim \sum_{\substack{n\in\mathbb{Z}\\ n\neq 1}} -\frac{i(-1)^n}{n-1}e^{inx}.}$$

At the endpoints, the periodic extension has one-sided values

$$f(\pi-) = -\pi, \qquad f(-\pi+) = \pi,$$

so the midpoint is 0:

$$\boxed{F(\pi) = F(-\pi) = 0.}$$

At $x = \pi/4$ the function is continuous, hence

$$\boxed{F(\pi/4) = \frac{\pi}{4}e^{i\pi/4} = \frac{\pi}{4\sqrt{2}}(1+i).}$$

Exercise 7

Let

$$f(x) = \begin{cases} 2 + \dfrac{2x}{\pi}, & -\pi < x < 0, \\ 2, & 0 < x < \pi. \end{cases}$$

(a) Compute the Fourier coefficients a_n and b_n.

(b) Define

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Determine the convergence of the series for every real x, and sketch g on $[-3\pi, 3\pi]$.

Solution. The constant coefficient is

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(2 + \frac{2x}{\pi}\right)dx + \int_0^{\pi} 2\,dx\right] = 3.$$

For $n \geq 1$, the constant pieces contribute zero to a_n, and

$$a_n = \frac{2}{\pi^2}\int_{-\pi}^{0} x\cos(nx)\,dx = \boxed{\frac{2(1-(-1)^n)}{\pi^2 n^2}.}$$

For b_n, the constant contributions cancel, while

$$b_n = \frac{2}{\pi^2}\int_{-\pi}^{0} x\sin(nx)\,dx = \boxed{-\frac{2(-1)^n}{\pi n}.}$$

Therefore

$$\boxed{f(x) \sim \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{2(1-(-1)^n)}{\pi^2 n^2}\cos(nx) - \frac{2(-1)^n}{\pi n}\sin(nx)\right].}$$

The series converges to the 2π-periodic extension of f at every continuity point. At $x = 0$ both one-sided limits equal 2, so $g(0) = 2$. At each odd multiple of π, the periodic extension jumps from 2 to 0, so the Fourier sum equals 1 there.

More explicitly, for each $k \in \mathbb{Z}$,

$$g(x) = \begin{cases} 2 + \dfrac{2(x-2k\pi)}{\pi}, & (2k-1)\pi < x < 2k\pi, \\ 2, & 2k\pi < x < (2k+1)\pi, \end{cases}$$

with

$$\boxed{g((2k+1)\pi) = 1, \qquad g(2k\pi) = 2.}$$

This description gives the requested sketch on $[-3\pi, 3\pi]$: alternating rising linear segments from 0 to 2 and horizontal segments at height 2, with midpoint value 1 at each jump.

Editorial notes

- In Exercise 2, the notation $\sum_{n=0}^{\infty} a_n$ is interpreted literally as including the full coefficient a_0. If the intended sum starts at $n = 1$, subtract a_0 from the displayed answers.
- In Exercise 4, $\lfloor x \rfloor$ denotes the floor function. The special value $1/2$ at integers agrees with the midpoint convergence value.

Chapter 2, Section 5
Fourier Series and Uniform Convergence

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Throughout, Fourier series are taken on $[-\pi, \pi]$ and extended 2π-periodically. We use the standard fact that if a periodic function is continuous and piecewise C^1, with only finitely many corners, then its Fourier coefficients are $O(n^{-2})$ and its Fourier series converges absolutely and uniformly. We also use the converse fact that a uniformly convergent Fourier series has a continuous periodic sum.

Exercise 1

Let

$$f(x) = \begin{cases} Ax + B, & -\pi \le x < 0, \\ \cos x, & 0 \le x \le \pi. \end{cases}$$

For which values of A and B does the Fourier series of f converge uniformly on $[-\pi, \pi]$?

Solution

Uniform convergence of the Fourier series implies that its sum is continuous and 2π-periodic. Since the Fourier series converges to f at every continuity point, f itself must match continuously at the joining point $x = 0$ and at the periodic endpoints $-\pi$ and π.

Continuity at $x = 0$ requires

$$B = f(0^-) = f(0^+) = \cos 0 = 1.$$

Thus

$$B = 1.$$

Periodicity requires

$$f(-\pi) = f(\pi).$$

Since

$$f(-\pi) = -A\pi + B, \qquad f(\pi) = \cos \pi = -1,$$

we obtain

$$-A\pi + B = -1.$$

Using $B = 1$ gives

$$-A\pi + 1 = -1, \qquad A = \frac{2}{\pi}.$$

Conversely, for

$$A = \frac{2}{\pi}, \qquad B = 1,$$

the function is continuous on the circle obtained by identifying $-\pi$ and π, and it is piecewise smooth. Therefore its Fourier coefficients decay like $O(n^{-2})$, so its Fourier series converges absolutely and uniformly.

Hence

$$\boxed{A = \frac{2}{\pi}, \qquad B = 1.}$$

Exercise 2

Let

$$g(x) = \begin{cases} \cos x, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases}$$

(a) Find the Fourier series of g on $[-\pi, \pi]$.

(b) Define

$$h(x) = \int_{-\pi}^{x} g(t)\,dt + a \sin \frac{x}{2}, \qquad -\pi \le x \le \pi.$$

For which values of a does the Fourier series of h converge uniformly on $[-\pi, \pi]$?

Solution to (a)

Write

$$g(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The constant coefficient is

$$a_0 = \frac{1}{\pi} \left(\int_{-\pi}^{0} \cos x\,dx + \int_{0}^{\pi} \sin x\,dx \right) = \frac{2}{\pi}.$$

Thus $a_0/2 = 1/\pi$.

For $n \ge 1$,

$$a_n = \frac{1}{\pi} \left(\int_{-\pi}^{0} \cos x \cos nx\,dx + \int_{0}^{\pi} \sin x \cos nx\,dx \right).$$

The first integral equals $\pi/2$ when $n = 1$ and 0 otherwise. The second is

$$\int_{0}^{\pi} \sin x \cos nx\,dx = \begin{cases} \dfrac{2}{1 - n^2}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Therefore

$$a_1 = \frac{1}{2}, \qquad a_{2k} = \frac{2}{\pi(1 - 4k^2)}, \qquad a_{2k+1} = 0 \quad (k \ge 1).$$

Similarly,

$$b_n = \frac{1}{\pi} \left(\int_{-\pi}^{0} \cos x \sin nx\,dx + \int_{0}^{\pi} \sin x \sin nx\,dx \right).$$

The second integral equals $\pi/2$ for $n = 1$ and vanishes otherwise. For even n,

$$\int_{-\pi}^{0} \cos x \sin nx\,dx = -\frac{2n}{n^2 - 1},$$

and it vanishes for odd n. Hence

$$b_1 = \frac{1}{2}, \qquad b_{2k} = -\frac{4k}{\pi(4k^2-1)}, \qquad b_{2k+1} = 0 \quad (k \geq 1).$$

Thus

$$\boxed{\begin{aligned} g(x) \sim{}& \frac{1}{\pi} + \frac{1}{2}(\cos x + \sin x) \\ &+ \sum_{k=1}^{\infty} \left[\frac{2}{\pi(1-4k^2)} \cos(2kx) - \frac{4k}{\pi(4k^2-1)} \sin(2kx) \right]. \end{aligned}}$$

At jump points, the series converges to the midpoint of the two one-sided limits.

Solution to (b)

The function h is continuous inside $(-\pi, \pi)$, because it is an integral of a piecewise continuous function plus a continuous term. Uniform convergence of its Fourier series on the closed interval requires the periodic endpoint condition

$$h(-\pi) = h(\pi).$$

We have

$$h(-\pi) = a \sin\left(-\frac{\pi}{2}\right) = -a.$$

Also,

$$\int_{-\pi}^{\pi} g(t)\,dt = \int_{-\pi}^{0} \cos t\,dt + \int_{0}^{\pi} \sin t\,dt = 2,$$

so

$$h(\pi) = 2 + a \sin\frac{\pi}{2} = 2 + a.$$

Therefore

$$-a = 2 + a, \qquad a = -1.$$

For $a = -1$, the periodic extension of h is continuous and piecewise C^1, so its Fourier series converges absolutely and uniformly.

Hence

$$\boxed{a = -1.}$$

Exercise 3

Let $a \notin \mathbb{Z}$. Prove that for $-\pi \leq x \leq \pi$,

$$\cos ax = \frac{\sin \pi a}{\pi a} + \sum_{n=1}^{\infty} (-1)^n \frac{2a \sin \pi a}{\pi(a^2 - n^2)} \cos nx,$$

and deduce that

$$\cot \pi a = \frac{1}{\pi} \left[\frac{1}{a} - \sum_{n=1}^{\infty} \frac{2a}{n^2 - a^2} \right].$$

Solution

The function $x \mapsto \cos ax$ is even, so all sine coefficients vanish. Its cosine coefficients are

$$a_0 = \frac{2}{\pi}\int_0^\pi \cos ax\,dx = \frac{2\sin \pi a}{\pi a}.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \cos ax \cos nx\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin((a-n)\pi)}{a-n} + \frac{\sin((a+n)\pi)}{a+n}\right]. \end{aligned}$$

Because $n \in \mathbb{Z}$,

$$\sin((a \pm n)\pi) = (-1)^n \sin \pi a.$$

Therefore

$$\begin{aligned} a_n &= \frac{(-1)^n \sin \pi a}{\pi}\left(\frac{1}{a-n} + \frac{1}{a+n}\right) \\ &= (-1)^n \frac{2a \sin \pi a}{\pi(a^2-n^2)}. \end{aligned}$$

This gives the required Fourier expansion.

Now set $x = \pi$. Since $\cos(n\pi) = (-1)^n$,

$$\cos \pi a = \frac{\sin \pi a}{\pi a} + \sum_{n=1}^{\infty} \frac{2a \sin \pi a}{\pi(a^2-n^2)}.$$

Since $a \notin \mathbb{Z}$, $\sin \pi a \neq 0$. Dividing by $\sin \pi a$ gives

$$\cot \pi a = \frac{1}{\pi a} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2a}{a^2-n^2}.$$

Equivalently,

$$\boxed{\cot \pi a = \frac{1}{\pi}\left[\frac{1}{a} - \sum_{n=1}^{\infty} \frac{2a}{n^2-a^2}\right].}$$

Exercise 4

Let f have Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Assume that there exist constants $c, d > 0$ such that

$$|a_n| \leq \frac{c}{n^2}, \qquad |b_n| \leq \frac{d}{n^2} \qquad (n \geq 1),$$

and that f is continuous on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$. Prove that the Fourier series of f converges uniformly to f.

Solution

For every real x,

$$|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n| \leq \frac{c+d}{n^2}.$$

Since

$$\sum_{n=1}^{\infty} \frac{c+d}{n^2} < \infty,$$

the Weierstrass M-test shows that the Fourier series converges absolutely and uniformly on $\mathbb{R}$. Let its sum be $S(x)$. Then S is continuous and 2π-periodic.

Uniform convergence permits term-by-term integration, so the Fourier coefficients of S are exactly a_n and b_n, the same as those of f. Set

$$q = f - S.$$

The assumptions $f(-\pi) = f(\pi)$ and continuity of f make q a continuous 2π-periodic function. Every Fourier coefficient of q is zero.

To conclude that $q \equiv 0$, use Fejér's theorem. The Fejér means of the Fourier series of a continuous periodic function converge uniformly to that function. Since all Fourier coefficients of q vanish, every Fejér mean of q is identically zero. Hence their uniform limit is zero, and therefore

$$q \equiv 0.$$

Thus $S = f$, and the original Fourier series converges uniformly to f.

Therefore

$$\boxed{\text{the Fourier series of } f \text{ converges uniformly to } f.}$$

Chapter 2, Section 6
Parseval Identities and Energy Calculations

Throughout, we use the standard real Fourier convention

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

for which Parseval's identity is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

For a complex Fourier series $f(x) \sim \sum_{n\in\mathbb{Z}} c_n e^{inx}$, we use

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx = 2\pi \sum_{n\in\mathbb{Z}} |c_n|^2.$$

Exercise 1

Compute

$$\int_{-\pi}^{\pi} \left|\sum_{n=1}^{\infty} \frac{1}{2^n} e^{inx}\right|^2 dx.$$

Solution. The complex Fourier coefficients are

$$c_n = \frac{1}{2^n} \quad (n \geq 1), \qquad c_n = 0 \quad (n \leq 0).$$

Hence Parseval's identity gives

$$\int_{-\pi}^{\pi} \left|\sum_{n=1}^{\infty} \frac{1}{2^n} e^{inx}\right|^2 dx = 2\pi \sum_{n=1}^{\infty} \frac{1}{4^n}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1/4}{1-1/4} = \frac{1}{3},$$

we obtain

$$\boxed{\int_{-\pi}^{\pi} \left|\sum_{n=1}^{\infty} \frac{1}{2^n} e^{inx}\right|^2 dx = \frac{2\pi}{3}.}$$

Exercise 2

Let f have Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Compute

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+\pi) - f(x)|^2 \, dx$$

in terms of a_n and b_n.

Solution. Using

$$\cos(n(x+\pi)) = (-1)^n \cos nx, \qquad \sin(n(x+\pi)) = (-1)^n \sin nx,$$

we find

$$f(x+\pi) - f(x) \sim \sum_{n=1}^{\infty} ((-1)^n - 1)(a_n \cos nx + b_n \sin nx).$$

For even n, the factor is zero; for odd n, it equals -2. Therefore the only nonzero Fourier coefficients of the difference are

$$-2a_n, \quad -2b_n \qquad (n \text{ odd}).$$

Parseval's identity yields

$$\boxed{\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+\pi) - f(x)|^2 \, dx = 4 \sum_{\substack{n \geq 1 \\ n \text{ odd}}} (a_n^2 + b_n^2).}$$

Equivalently,

$$\boxed{4 \sum_{k=0}^{\infty} \left(a_{2k+1}^2 + b_{2k+1}^2 \right).}$$

Exercise 3

For each positive integer n, define

$$f_n(x) = 1 + \sum_{k=1}^{n} (\cos kx - \sin kx).$$

Compute

$$\int_{-\pi}^{\pi} |f_n(x)|^2 \, dx.$$

Solution. In the standard real Fourier notation,

$$\frac{a_0}{2} = 1 \quad \Longrightarrow \quad a_0 = 2,$$

and for $1 \leq k \leq n$,

$$a_k = 1, \qquad b_k = -1.$$

All other coefficients vanish. Therefore Parseval gives

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f_n(x)|^2 \, dx = \frac{2^2}{2} + \sum_{k=1}^{n} (1^2 + (-1)^2) = 2 + 2n.$$

Thus

$$\boxed{\int_{-\pi}^{\pi} |f_n(x)|^2 \, dx = 2\pi(n+1).}$$

Exercise 4

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series of f on $[-\pi, \pi]$. Compute

$$I = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

when

$$f(x) = e^{-x}.$$

Solution. The quantity I contains only the constant and cosine coefficients. These are precisely the Fourier coefficients of the even part of f:

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x.$$

Therefore Parseval applied to f_e gives

$$I = \frac{1}{\pi} \int_{-\pi}^{\pi} \cosh^2 x \, dx.$$

Using

$$\cosh^2 x = \frac{1 + \cosh 2x}{2},$$

we obtain

$$\int_{-\pi}^{\pi} \cosh^2 x \, dx = \int_{-\pi}^{\pi} \frac{1 + \cosh 2x}{2} \, dx = \pi + \frac{1}{2} \sinh(2\pi).$$

Hence

$$\boxed{I = 1 + \frac{\sinh(2\pi)}{2\pi}.}$$

For comparison, the cosine coefficients may also be computed directly:

$$a_0 = \frac{2 \sinh \pi}{\pi}, \qquad a_n = \frac{2(-1)^n \sinh \pi}{\pi(1 + n^2)} \quad (n \geq 1),$$

and substitution into the definition of I gives the same result.

Source: Section 6 exercise sheet supplied by the user. Notation has been translated and normalized, while the mathematical content has been preserved.

Chapter 2, Section 7
Half-Range Fourier Series

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Throughout, Fourier series are interpreted in the standard Dirichlet sense: at a jump, the series converges to the average of the two one-sided limits.

Exercise 1: Odd extension of a triangular function

Let

$$f(x) = \begin{cases} x, & 0 \le x \le \pi/2, \\ \pi - x, & \pi/2 \le x \le \pi, \end{cases}$$

and let $\widetilde{f}$ be the odd extension of f to $[-\pi, \pi]$. Find the Fourier series of $\widetilde{f}$.

Solution

Because $\widetilde{f}$ is odd, its Fourier series contains only sine terms:

$$\widetilde{f}(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \qquad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx.$$

Split the integral at $\pi/2$:

$$I_n := \int_0^{\pi} f(x) \sin(nx)\, dx = \int_0^{\pi/2} x \sin(nx)\, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin(nx)\, dx.$$

In the second integral put $u = \pi - x$. Since

$$\sin(n(\pi - u)) = -(-1)^n \sin(nu),$$

we obtain

$$I_n = \left(1 - (-1)^n\right) \int_0^{\pi/2} x \sin(nx)\, dx.$$

Integration by parts gives

$$\int_0^{\pi/2} x \sin(nx)\, dx = -\frac{\pi}{2n} \cos\frac{n\pi}{2} + \frac{1}{n^2} \sin\frac{n\pi}{2}.$$

For even n, $I_n = 0$. For odd n, the cosine term vanishes and therefore

$$I_n = \frac{2}{n^2} \sin\frac{n\pi}{2}.$$

Thus, in a formula valid for every $n \ge 1$,

$$\boxed{b_n = \frac{4}{\pi n^2} \sin\frac{n\pi}{2}}.$$

Equivalently,

$$\boxed{\widetilde{f}(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)x)}$$

at every point, because the odd periodic extension is continuous and piecewise smooth.

Exercise 2: Combining the half-range sine and cosine series

Let f belong to the usual piecewise-regular Fourier class on $[0, \pi]$. Suppose

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

is its half-range sine series and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

is its half-range cosine series. Determine

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big)$$

for $x \in [-\pi, \pi]$.

Solution

The cosine series is the Fourier series of the even extension

$$f_e(x) = f(|x|),$$

whereas the sine series is the Fourier series of the odd extension

$$f_o(x) = \begin{cases} f(x), & x > 0, \\ -f(-x), & x < 0. \end{cases}$$

Consequently, g is the Fourier series of $f_e + f_o$.

For $0 < x < \pi$, both extensions have the same one-sided midpoint value as f, so

$$\boxed{g(x) = f(x^-) + f(x^+)}.$$

In particular, if f is continuous at x, then $g(x) = 2f(x)$.

For $-\pi < x < 0$, the midpoint values of the even and odd extensions cancel, hence

$$\boxed{g(x) = 0}.$$

At the joining point and periodic endpoints,

$$\boxed{g(0) = f(0^+), \qquad g(\pi) = g(-\pi) = f(\pi^-)}.$$

Thus, when f is continuous on $[0, \pi]$,

$$\boxed{g(x) = \begin{cases} 0, & -\pi < x < 0, \\ f(0), & x = 0, \\ 2f(x), & 0 < x < \pi, \\ f(\pi), & x = \pm\pi. \end{cases}}$$

Exercise 3: The sine series of $x(\pi - x)$ and two sums

(a) Find the sine series of $f(x) = x(\pi - x)$ on $[0, \pi]$.

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

(c) Prove that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

Solution to (a)

The sine coefficients are

$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)\,dx.$$

Two integrations by parts give

$$\int_0^\pi x(\pi - x)\sin(nx)\,dx = \frac{2(1-(-1)^n)}{n^3}.$$

Therefore

$$b_n = \frac{4(1-(-1)^n)}{\pi n^3}.$$

Thus $b_n = 0$ for even n, while $b_n = 8/(\pi n^3)$ for odd n. Hence

$$\boxed{x(\pi - x) = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{(2k+1)^3}, \qquad 0 \le x \le \pi.}$$

Solution to (b)

Parseval's identity for a sine series on $[0, \pi]$ states that

$$\int_0^\pi |f(x)|^2\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} b_n^2.$$

Here

$$\int_0^\pi x^2(\pi - x)^2\,dx = \frac{\pi^5}{30}.$$

Using the coefficients above,

$$\frac{\pi^5}{30} = \frac{\pi}{2}\frac{64}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6}.$$

Therefore

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6} = \frac{\pi^6}{960}.$$

If $S = \sum_{n=1}^{\infty} n^{-6}$, then the even terms contribute $S/2^6 = S/64$, so

$$\frac{63}{64}S = \frac{\pi^6}{960}.$$

Hence

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}}.$$

Solution to (c)

Set $x = \pi/2$ in the sine series. Since

$$\sin\frac{(2k+1)\pi}{2} = (-1)^k, \qquad f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4},$$

we obtain

$$\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}.$$

Thus

$$\boxed{1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.}$$

Exercise 4: The half-range sine series of $\cos x$

(a) Find the sine series

$$\cos x \sim \sum_{n=1}^{\infty} b_n \sin(nx), \qquad 0 \le x \le \pi.$$

(b) Does the resulting series converge at every point of $[-\pi, \pi]$?

(c) Define

$$F(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Sketch the graph of F on $[-\pi, \pi]$.

Solution to (a)

We have

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin(nx)\, dx.$$

Using

$$\sin(nx)\cos x = \frac{1}{2}\big(\sin((n+1)x) + \sin((n-1)x)\big),$$

one finds that $b_n = 0$ for odd n (including $n = 1$), while for even $n \ge 2$,

$$b_n = \frac{2}{\pi}\left(\frac{1}{n-1} + \frac{1}{n+1}\right) = \frac{4n}{\pi(n^2-1)}.$$

Therefore

$$\boxed{\cos x = \sum_{k=1}^{\infty} \frac{8k}{\pi(4k^2-1)} \sin(2kx), \qquad 0 < x < \pi.}$$

Solution to (b)

This is the Fourier series of the odd 2π-periodic extension of $\cos x$ from $(0, \pi)$. That extension is piecewise smooth, so its Fourier series converges at every real point. At a jump it converges to the midpoint of the one-sided limits.

Solution to (c)

On the open intervals,

$$F(x) = \begin{cases} -\cos x, & -\pi < x < 0, \\ \cos x, & 0 < x < \pi. \end{cases}$$

At $x = 0$, the one-sided limits are -1 and 1, so $F(0) = 0$. At the periodic endpoints $\pm\pi$, the one-sided periodic limits are -1 and 1, so again the midpoint is zero. Thus

$$\boxed{F(x) = \begin{cases} -\cos x, & -\pi < x < 0, \\ 0, & x = 0, \\ \cos x, & 0 < x < \pi, \\ 0, & x = \pm\pi. \end{cases}}$$

The graph consists of the reflected negative cosine arc on $(-\pi, 0)$ and the ordinary cosine arc on $(0, \pi)$, with midpoint values 0 at the three jump representatives $-\pi, 0, \pi$.

Prepared as an English worked-solution companion to the supplied Section 7 exercise sheet.

Chapter 2, Section 8
Integrated Fourier Series and Coefficient Decay

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Conventions

For a 2π-periodic function on $[-\pi, \pi]$ we use

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

At a jump, the Fourier series is understood to converge to the midpoint of the one-sided limits.

Exercise 1

Let $f(x) = |x|$ on $[-\pi, \pi]$.

(a) Fourier coefficients

The function is even, hence $b_n = 0$. Also

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2}.$$

Thus

$$a_{2k} = 0, \qquad a_{2k-1} = -\frac{4}{\pi(2k-1)^2},$$

and therefore

$$\boxed{|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)^2}.}$$

(b) Convergence of $\sum na_n \sin nx$

Using the coefficients above,

$$\sum_{n=1}^{\infty} na_n \sin nx = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{2k-1}.$$

For $x \notin \pi\mathbb{Z}$, the partial sums of $\sin((2k-1)x)$ are bounded and $1/(2k-1) \downarrow 0$, so Dirichlet's test gives convergence. If $x \in \pi\mathbb{Z}$, every term is zero. Hence the series converges for every real x.

(c) The graph of g

Define

$$g(x) = -\sum_{n=1}^{\infty} n a_n \sin nx = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{2k-1}.$$

This is the Fourier series of the 2π-periodic square wave. Hence

$$\boxed{g(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \\ 0, & x \in \pi\mathbb{Z}, \end{cases} \qquad g(x+2\pi) = g(x).}$$

On $[-2\pi, 2\pi]$ it alternates between -1 and 1, with value 0 at $-2\pi, -\pi, 0, \pi, 2\pi$.

(d) Two sums

At $x = 0$ in the Fourier series of $|x|$,

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2},$$

so

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.}$$

Parseval gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2.$$

Thus

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4},$$

and therefore

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}.}$$

Exercise 2

Let f be even and satisfy $\int_{-\pi}^{\pi} f(t)\,dt = 5$. Define

$$F(x) = \int_{-\pi}^{x} f(t)\,dt.$$

Because f is even,

$$F(0) = \int_{-\pi}^{0} f(t)\,dt = \frac{1}{2}\int_{-\pi}^{\pi} f(t)\,dt = \frac{5}{2}.$$

Also $F(-\pi) = 0$ and $F(\pi) = 5$. The Fourier series G of F converges at interior continuity points to F, while at the identified endpoints it converges to the average of the endpoint limits. Hence

$$G(0) = F(0) = \frac{5}{2},$$

and

$$G(-\pi) = G(\pi) = \frac{F(-\pi+) + F(\pi-)}{2} = \frac{0+5}{2} = \frac{5}{2}.$$

Therefore

$$\boxed{G(-\pi) = G(0) = G(\pi) = \frac{5}{2}.}$$

Exercise 3

Let

$$f(x)=\begin{cases}\dfrac{\pi}{4}, & -\pi<x<0,\\ \dfrac{\pi}{4}-x, & 0<x<\pi.\end{cases}$$

(a) Fourier coefficients

A direct calculation gives

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=0.$$

Moreover,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1-(-1)^n}{\pi n^2},$$

and

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=\frac{(-1)^n}{n}.$$

Thus

$$\boxed{a_0=0,\qquad a_n=\frac{1-(-1)^n}{\pi n^2},\qquad b_n=\frac{(-1)^n}{n}.}$$

(b) Evaluation of $S(x)$

Define

$$S(x)=\frac{a_0x}{2}+\sum_{n=1}^{\infty}\frac{1}{n}\big(a_n\sin nx-b_n\cos nx\big).$$

Termwise differentiation gives the Fourier series of f, so $S'=f$ away from the jump points. Hence S is an antiderivative of f, up to a constant. A continuous periodic antiderivative is

$$H(x)=\begin{cases}\dfrac{\pi x}{4}+C, & -\pi<x<0,\\ \dfrac{\pi x}{4}-\dfrac{x^2}{2}+C, & 0<x<\pi.\end{cases}$$

Since the displayed series for S has zero mean, choose C so that H has zero mean. With $C=0$, its mean is $-\pi^2/12$, hence $C=\pi^2/12$. Therefore

$$\boxed{S(x)=\begin{cases}\dfrac{\pi^2}{12}+\dfrac{\pi x}{4}, & -\pi<x<0,\\ \dfrac{\pi^2}{12}+\dfrac{\pi x}{4}-\dfrac{x^2}{2}, & 0<x<\pi.\end{cases}}$$

The two formulas agree at $x=0$, and the endpoint limits agree as required by periodicity.

Exercise 4

Let f be piecewise continuous, 2π-periodic, and satisfy

$$\int_{-\pi}^{\pi} f(x)\,dx=0.$$

Define $g(x)=\int_0^x f(t)\,dt$.

(a) Periodicity

For every x,

$$g(x+2\pi)-g(x)=\int_x^{x+2\pi} f(t)\,dt=\int_{-\pi}^{\pi} f(t)\,dt=0.$$

Hence

$$\boxed{g(x+2\pi)=g(x).}$$

(b) Fourier coefficients

Let

$$f(x)\sim\sum_{n\in\mathbb{Z}} c_n e^{inx},\qquad g(x)\sim\sum_{n\in\mathbb{Z}} d_n e^{inx}.$$

For $n\neq 0$, integration by parts and $g(\pi)=g(-\pi)$ give

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} g'(x)e^{-inx}\,dx=in\,d_n.$$

Therefore

$$\boxed{d_n=\frac{c_n}{in}\qquad (n\neq 0).}$$

The constant coefficient is

$$d_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx,$$

which depends on the normalization $g(0)=0$ and is not determined by the relation above. Since g is continuous and piecewise C^1, its Fourier series converges to $g(x)$ for every real x.

Editorial note. The source statement appears to write $d_n=c_n/(in)$ for every integer n. The correct statement excludes $n=0$.

Exercise 5

(a) Integrating a full Fourier series

Suppose

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

For $-\pi\le c,x\le\pi$ define

$$H(x)=\frac{a_0}{2}(x-c)+\sum_{n=1}^{\infty}\left[\frac{a_n}{n}(\sin nx-\sin nc)-\frac{b_n}{n}(\cos nx-\cos nc)\right].$$

By Bessel's inequality, (a_n) and (b_n) belong to ℓ^2. Hence, by Cauchy–Schwarz,

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n}<\infty,\qquad \sum_{n=1}^{\infty}\frac{|b_n|}{n}<\infty.$$

Thus the series defining H converges uniformly by the Weierstrass test. Its Fourier coefficients are exactly those of $x\mapsto\int_c^x f(t)\,dt$, and both functions vanish at $x=c$. Therefore

$$\boxed{\int_c^x f(t)\,dt=\frac{a_0}{2}(x-c)+\sum_{n=1}^{\infty}\left[\frac{a_n}{n}(\sin nx-\sin nc)-\frac{b_n}{n}(\cos nx-\cos nc)\right],}$$

with uniform convergence on $[-\pi,\pi]$.

(b) Integrating a cosine series

If on $[0, \pi]$

$$g(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

apply part (a) to the even extension and take $c = 0$. Then

$$\boxed{\int_0^x g(t)\,dt = \frac{a_0 x}{2} + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx,}$$

uniformly on $[0, \pi]$.

(c) Integrating a sine series

If

$$h(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \quad (0 \le x \le \pi),$$

apply part (a) to the odd extension. We obtain

$$\int_0^x h(t)\,dt = \sum_{n=1}^{\infty} \frac{b_n}{n}(1 - \cos nx).$$

Since $\sum |b_n|/n < \infty$, put

$$\beta = \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

Then

$$\boxed{\int_0^x h(t)\,dt = \beta - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos nx,}$$

and the convergence is uniform on $[0, \pi]$.

Exercise 6

Assume f is continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and f' is piecewise continuous. Integration by parts gives

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = -\frac{1}{\pi n} \int_{-\pi}^{\pi} f'(x) \sin nx\,dx,$$

because the sine boundary term vanishes. Hence, by the Riemann–Lebesgue lemma,

$$\boxed{n a_n \longrightarrow 0.}$$

Similarly,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx = \frac{1}{\pi n} \int_{-\pi}^{\pi} f'(x) \cos nx\,dx,$$

where the boundary term vanishes because $f(-\pi) = f(\pi)$. Thus

$$\boxed{n b_n \longrightarrow 0.}$$

The stronger assertion with n^2 is false. Take $f(x) = |x|$. It satisfies all the hypotheses and

$$a_{2k} = 0, \qquad a_{2k-1} = -\frac{4}{\pi(2k-1)^2}, \qquad b_n = 0.$$

Therefore $n^2 a_n$ equals 0 along the even indices and $-4/\pi$ along the odd indices, so it does not even have a limit. Hence

$$\boxed{n^2 a_n \to 0 \text{ and } n^2 b_n \to 0 \text{ need not hold.}}$$

Chapter 2, Section 9
Fourier Series: Changes of Interval, Periodicity, and Parseval

Conventions

For a 2π-periodic Fourier series on $[-\pi, \pi]$ we use

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big),$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

At a jump, the Fourier series converges to the midpoint of the two one-sided limits.

Exercise 1

Let $f(x) = x^2$. Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big)$$

be its Fourier series when f is prescribed on $[-\pi, \pi]$, and let

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} \big(A_n \cos nx + B_n \sin nx\big)$$

be its Fourier series when f is prescribed on $[0, 2\pi]$. Define

$$h(x) = \frac{a_0 - A_0}{2} + \sum_{n=1}^{\infty} \big((a_n - A_n)\cos nx + (b_n - B_n)\sin nx\big).$$

Find $h(x)$ for $-\pi \le x \le \pi$, and draw its graph on $[-\pi, 2\pi]$.

Solution. Let S_1 denote the first Fourier series and S_2 the second. Their difference is exactly $h = S_1 - S_2$. The first series represents the 2π-periodic extension of x^2 from $[-\pi, \pi]$. Since the endpoint values agree, this extension is continuous, and

$$S_1(x) = x^2 \qquad (-\pi \le x \le \pi).$$

The second series represents the 2π-periodic extension of x^2 from $[0, 2\pi]$. For $-\pi < x < 0$, the corresponding point in $[0, 2\pi]$ is $x + 2\pi$, so

$$S_2(x) = (x + 2\pi)^2.$$

For $0 < x < \pi$, we have $S_2(x) = x^2$. At $x = 0$, the function is continuous from within the interval and $S_2(0) = 0$. At $x = \pm\pi$, both periodic representatives have the value π^2.

Therefore

$$\boxed{h(x) = \begin{cases} -4\pi x - 4\pi^2, & -\pi < x < 0, \\ 0, & x = 0 \text{ or } 0 < x \le \pi, \end{cases}}$$

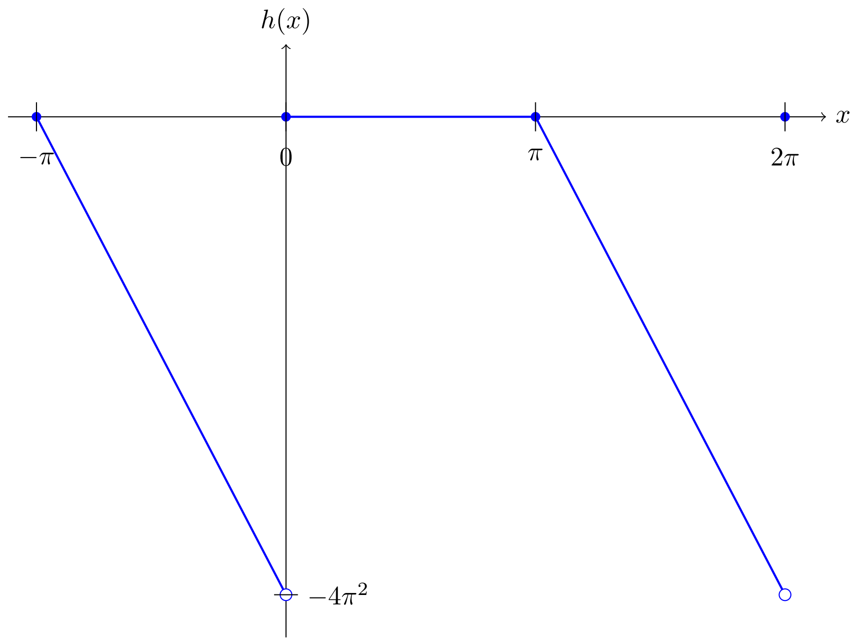
with $h(-\pi) = 0$. Equivalently, on $[-\pi, \pi]$,

$$h(x) = \begin{cases} -4\pi(x+\pi), & -\pi < x < 0, \\ 0, & x \in \{-\pi\} \cup [0, \pi]. \end{cases}$$

Since h is represented by a 2π-periodic trigonometric series, it is 2π-periodic. Hence on $[-\pi, 2\pi]$,

$$h(x) = \begin{cases} -4\pi(x+\pi), & -\pi < x < 0, \\ 0, & 0 \le x \le \pi, \\ -4\pi(x-\pi), & \pi < x < 2\pi, \end{cases}$$

and $h(-\pi) = h(0) = h(\pi) = h(2\pi) = 0$.



Exercise 2

Let $f : \mathbb{R} \to \mathbb{C}$ be piecewise continuous and π-periodic. Suppose its Fourier series on $[0, \pi]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos 2nx + b_n \sin 2nx\big),$$

and its Fourier series on $[-\pi, \pi]$ is

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \big(A_n \cos nx + B_n \sin nx\big).$$

Express A_n, B_n in terms of a_n, b_n.

Solution. Because f has period π, only even harmonics can appear in its 2π-Fourier series.

For $n \geq 0$,

$$A_{2n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(2nx) \, dxx.$$

Split the integral into two intervals of length π and use $f(x + \pi) = f(x)$ together with

$$\cos(2n(x + \pi)) = \cos(2nx), \qquad \sin(2n(x + \pi)) = \sin(2nx).$$

Then

$$A_{2n} = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(2nx) \, dxx = a_n,$$

and similarly

$$B_{2n} = b_n.$$

For odd indices,

$$\cos((2n + 1)(x + \pi)) = -\cos((2n + 1)x),$$

with the same sign reversal for sine. The two half-period integrals therefore cancel, giving

$$A_{2n+1} = B_{2n+1} = 0.$$

Thus

$$\boxed{A_0 = a_0, \qquad A_{2n} = a_n, \qquad B_{2n} = b_n, \qquad A_{2n+1} = B_{2n+1} = 0.}$$

Exercise 3

Let

$$f(t) = \begin{cases} A \sin(\omega_0 t), & 0 < t < T/2, \\ 0, & T/2 \leq t < T, \end{cases} \qquad \omega_0 = \frac{2\pi}{T}.$$

Find the complex Fourier series of f on $[0, T]$.

Solution. Write

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \qquad c_n = \frac{1}{T} \int_{0}^{T} f(t) e^{-in\omega_0 t} \, dxt.$$

Since f vanishes on the second half-period,

$$c_n = \frac{A}{T} \int_{0}^{T/2} \sin(\omega_0 t) e^{-in\omega_0 t} \, dxt.$$

With $u = \omega_0 t$, so that $dxt = dxu/\omega_0$ and $0 < u < \pi$,

$$c_n = \frac{A}{2\pi} \int_{0}^{\pi} \sin u \, e^{-inu} \, dxu.$$

For $n \neq \pm 1$, direct integration gives

$$\int_{0}^{\pi} \sin u \, e^{-inu} \, dxu = -\frac{1 + (-1)^n}{n^2 - 1}.$$

At the exceptional indices,

$$\int_{0}^{\pi} \sin u \, e^{-iu} \, dxu = -\frac{i\pi}{2}, \qquad \int_{0}^{\pi} \sin u \, e^{iu} \, dxu = \frac{i\pi}{2}.$$

Therefore

$$\boxed{c_n = \begin{cases} -\dfrac{A\bigl(1 + (-1)^n\bigr)}{2\pi(n^2 - 1)}, & n \neq \pm 1, \\ -\dfrac{iA}{4}, & n = 1, \\ \dfrac{iA}{4}, & n = -1. \end{cases}}$$

Equivalently,

$$c_0 = \frac{A}{\pi}, \qquad c_{2k} = -\frac{A}{\pi(4k^2-1)} \quad (k \neq 0),$$

while $c_n = 0$ for odd $|n| \geq 3$, and $c_{\pm 1} = \mp iA/4$. Thus

$$\boxed{f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}.}$$

At the discontinuity points, the series converges to the midpoint of the one-sided limits.

Exercise 4

Let $f \in E[a,b]$. State Parseval's identity for the Fourier expansion of f on $[a,b]$ and explain why it is valid.

Solution. Let $L = b - a$ and use the complex Fourier basis

$$\phi_n(x) = e^{2\pi in(x-a)/L}, \qquad n \in \mathbb{Z}.$$

The coefficients are

$$c_n = \frac{1}{L}\int_a^b f(x)e^{-2\pi in(x-a)/L}\,dxx.$$

Parseval's identity is

$$\boxed{\frac{1}{L}\int_a^b |f(x)|^2\,dxx = \sum_{n=-\infty}^{\infty} |c_n|^2.}$$

In real form, if

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi n(x-a)}{L} + b_n \sin\frac{2\pi n(x-a)}{L}\right],$$

where

$$a_n = \frac{2}{L}\int_a^b f(x)\cos\frac{2\pi n(x-a)}{L}\,dxx, \quad b_n = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi n(x-a)}{L}\,dxx,$$

then

$$\boxed{\frac{1}{L}\int_a^b |f(x)|^2\,dxx = \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\big(|a_n|^2 + |b_n|^2\big).}$$

For real-valued f, the absolute values on a_n, b_n may be omitted.

To justify the identity, make the affine change of variable

$$x = a + \frac{L}{2\pi}(t+\pi), \qquad -\pi \leq t \leq \pi.$$

This carries the interval $[a,b]$ to $[-\pi,\pi]$ and transforms the above trigonometric system into the standard Fourier system. Parseval's identity on $[-\pi,\pi]$ then gives the stated formula after accounting for the constant Jacobian $L/(2\pi)$.

Exercise 5

Let $f \in E[-\pi,\pi]$ have Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n \cos nx + b_n \sin nx\big).$$

Assume that f is periodic with period π/m, where $m \in \mathbb{N}$ is fixed. Prove that

$$a_n = b_n = 0$$

whenever n is not divisible by $2m$.

Solution. It is most efficient to use complex Fourier coefficients

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dxx.$$

Because f has period π/m, shifting the variable by π/m does not change the integral. Hence

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(x + \frac{\pi}{m}\right) e^{-inx}\, dxx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-in(x-\pi/m)}\, dxx \\ &= e^{in\pi/m} c_n. \end{aligned}$$

Therefore

$$\left(1 - e^{in\pi/m}\right) c_n = 0.$$

Now

$$e^{in\pi/m} = 1 \quad \Longleftrightarrow \quad \frac{n\pi}{m} = 2\pi k \quad \Longleftrightarrow \quad n = 2mk$$

for some $k \in \mathbb{Z}$. Thus $c_n = 0$ unless $2m$ divides n.

The real Fourier coefficients satisfy, for $n \geq 1$,

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n}).$$

If $2m \nmid n$, then both c_n and c_{-n} vanish, and consequently

$$\boxed{a_n = b_n = 0 \qquad \text{whenever } 2m \nmid n.}$$

This expresses the expected frequency principle: a function with period π/m has fundamental angular frequency $2m$, so only integer multiples of $2m$ can occur.

Editorial note

In Exercise 5, the printed period is π/m. This is consistent with the stated conclusion that only indices divisible by $2m$ survive. Had the period been $2\pi/m$, the surviving indices would instead be the multiples of m.

Chapter 2 – General Review Exercises
Fourier Series: General Review Exercises

This document was generated by **GPT-5.6 Sol** (extended) which translated and solved the exercises in the supplied section in our **Fourier Series and Laplace Transforms** course textbook (Hebrew): https://www.samyzaf.com/technion/fourier/fourier.pdf
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The normalization used throughout is stated separately in each exercise. At jump points, Fourier series are interpreted in the usual Dirichlet sense, namely as the average of the one-sided limits.

Exercise 1

Let $f(x) = x + \cos x$ on $[0, 4\pi]$, and let

$$f(x) \underset{\text{FS}}{\sim} \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{nx}{2} + b_n \sin \frac{nx}{2} \right).$$

(a) Compute a_n and b_n.

(b) For

$$g(x) = \frac{A_0}{2} + \sum_{n=1}^{5} B_n \sin \frac{nx}{2},$$

find the coefficients that minimize the $L^2(0, 4\pi)$ distance between f and g.

Solution. For a function on an interval of length 4π, the coefficients are

$$a_n = \frac{1}{2\pi} \int_0^{4\pi} f(x) \cos \frac{nx}{2}\, dx, \qquad b_n = \frac{1}{2\pi} \int_0^{4\pi} f(x) \sin \frac{nx}{2}\, dx.$$

First,

$$a_0 = \frac{1}{2\pi} \int_0^{4\pi} (x + \cos x)\, dx = 4\pi.$$

For $n \geq 1$, integration by parts gives

$$\frac{1}{2\pi} \int_0^{4\pi} x \cos \frac{nx}{2}\, dx = 0, \qquad \frac{1}{2\pi} \int_0^{4\pi} x \sin \frac{nx}{2}\, dx = -\frac{4}{n}.$$

Orthogonality gives

$$\frac{1}{2\pi} \int_0^{4\pi} \cos x \cos \frac{nx}{2}\, dx = \begin{cases} 1, & n = 2, \\ 0, & n \neq 2, \end{cases} \qquad \int_0^{4\pi} \cos x \sin \frac{nx}{2}\, dx = 0.$$

Hence

$$\boxed{a_0 = 4\pi, \qquad a_n = \begin{cases} 1, & n = 2, \\ 0, & n \neq 2, \end{cases} \qquad b_n = -\frac{4}{n}.}$$

The best approximation from the subspace spanned by $1, \sin(x/2), \ldots, \sin(5x/2)$ is the orthogonal projection of f onto that subspace. Therefore its coefficients are precisely the corresponding Fourier coefficients:

$$\boxed{A_0 = 4\pi, \qquad B_n = -\frac{4}{n} \quad (1 \leq n \leq 5).}$$

Exercise 2

Let $f(x) = x^2$ for $\pi \le x \le 3\pi$, with Fourier series

$$f(x) \underset{\text{FS}}{\sim} \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

(a) Compute a_n, b_n.

(b) On $[-\pi, \pi]$, compute and sketch

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{nx}{2}.$$

(c) On $[-\pi, \pi]$, compute and sketch

$$h(x) = \sum_{n=1}^{\infty} b_n \sin \frac{nx}{2}.$$

Solution. Put $y = x - 2\pi$. Then $y \in [-\pi, \pi]$ and

$$x^2 = (y + 2\pi)^2 = y^2 + 4\pi y + 4\pi^2.$$

Since $\cos(nx) = \cos(ny)$ and $\sin(nx) = \sin(ny)$,

$$\boxed{a_0 = \frac{26\pi^2}{3}, \qquad a_n = \frac{4(-1)^n}{n^2}, \qquad b_n = \frac{8\pi(-1)^{n+1}}{n}.}$$

Indeed, these are the familiar Fourier coefficients of y^2, $4\pi y$, and the constant $4\pi^2$.

Let $t = x/2$. The standard identity

$$t^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt, \qquad |t| \le \pi,$$

gives, for $|x| \le \pi$,

$$\boxed{g(x) = 4\pi^2 + \frac{x^2}{4}.}$$

Similarly,

$$4\pi t = 8\pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt, \qquad |t| < \pi,$$

so

$$\boxed{h(x) = 2\pi x \qquad (-\pi \le x \le \pi).}$$

Thus g is an upward-opening parabola with vertex $(0, 4\pi^2)$, while h is the straight line of slope 2π through the origin.

Exercise 3

Let $f(x) = |\sin x|$ on $[-\pi, \pi]$.

(a) Find its real Fourier series.

(b) Let

$$g(x) = \sum_{n=1}^{\infty} \big(-na_n \sin nx + nb_n \cos nx\big).$$

Determine and sketch g on $[-\pi, \pi]$.

(c) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}, \quad \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}, \quad \sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}.$$

Solution. The function is even, so $b_n = 0$. A direct computation gives

$$\boxed{|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2-1}.}$$

Equivalently, $a_0 = 4/\pi$, $a_{2n} = -4/[\pi(4n^2-1)]$, and $a_{2n-1} = 0$.

The series defining g is the termwise derivative of the Fourier series. Hence, away from the corners of $|\sin x|$,

$$g(x) = f'(x) = \begin{cases} -\cos x, & -\pi < x < 0, \\ \cos x, & 0 < x < \pi. \end{cases}$$

At $x = -\pi, 0, \pi$, the derivative series converges to the average of the two one-sided limits, namely 0. Thus

$$\boxed{g(x) = \operatorname{sgn}(\sin x) \cos x \text{ at smooth points}, \qquad g(-\pi) = g(0) = g(\pi) = 0.}$$

Set $x = 0$ in the Fourier series:

$$0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1},$$

so

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.}$$

Set $x = \pi/2$:

$$1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1},$$

therefore

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{2-\pi}{4}.}$$

Parseval for f gives

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x|^2 \, dxx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2.$$

Since the left-hand side is 1,

$$1 = \frac{8}{\pi^2} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2},$$

and hence

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2-8}{16}.}$$

Finally, Parseval applied to $g = f'$ (almost everywhere) yields

$$1 = \frac{1}{\pi} \int_{-\pi}^{\pi} |g(x)|^2 \, dxx = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2},$$

so

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{64}.}$$

Exercise 4

Find the complex Fourier series on $[-\pi, \pi]$ of

$$f(t) = \frac{1}{1 - \frac{1}{2}e^{-it}}.$$

Solution. Since $|\frac{1}{2}e^{-it}| = 1/2 < 1$, the geometric expansion is uniformly convergent:

$$f(t) = \sum_{k=0}^{\infty} \left(\frac{1}{2}e^{-it}\right)^k = \sum_{k=0}^{\infty} 2^{-k} e^{-ikt}.$$

Thus, in the convention $f(t) \underset{\text{FS}}{\sim} \sum_{n\in\mathbb{Z}} c_n e^{int}$,

$$\boxed{c_n = \begin{cases} 2^n, & n \le 0, \\ 0, & n > 0. \end{cases}}$$

Exercise 5

Let $f : \mathbb{R} \to \mathbb{C}$ be piecewise continuous and 2π-periodic, with

$$f(x) \underset{\text{FS}}{\sim} \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Define

$$g(x) = \frac{f(x) + f(x+\pi)}{2}, \qquad g(x) \underset{\text{FS}}{\sim} \sum_{n=-\infty}^{\infty} d_n e^{inx}.$$

(a) Express d_n in terms of c_n.

(b) If $f(x) = x$ for $-\pi < x < \pi$, evaluate $\sum_{n=1}^{\infty} |c_{2n}|^2$.

Solution. Because

$$f(x+\pi) \underset{\text{FS}}{\sim} \sum_{n\in\mathbb{Z}} c_n e^{in(x+\pi)} = \sum_{n\in\mathbb{Z}} (-1)^n c_n e^{inx},$$

we obtain

$$\boxed{d_n = \frac{1+(-1)^n}{2} c_n = \begin{cases} c_n, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}}$$

For $f(x) = x$,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx x = \frac{i(-1)^n}{n} \qquad (n \neq 0),$$

and $c_0 = 0$. Therefore

$$\sum_{n=1}^{\infty} |c_{2n}|^2 = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \cdot \frac{\pi^2}{6}.$$

Hence

$$\boxed{\sum_{n=1}^{\infty} |c_{2n}|^2 = \frac{\pi^2}{24}.}$$

Exercise 6

Assume $f(x+\pi) = f(x)$ for all real x, and on $[0,\pi]$ let

$$f(x) = \begin{cases} \sin 2x, & 0 \le x \le \pi/2, \\ 0, & \pi/2 \le x \le \pi. \end{cases}$$

(a) Prove that

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin 2x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 4nx}{(2n-1)(2n+1)},$$

and deduce

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2} = \frac{\pi^2-8}{16}.$$

(b) Evaluate, for $0 \le x \le \pi$,

$$\frac{\sin 4x}{1\cdot 2\cdot 3} + \frac{\sin 8x}{3\cdot 4\cdot 5} + \frac{\sin 12x}{5\cdot 6\cdot 7} + \cdots .$$

Solution. The function has period π, but we may regard it as 2π-periodic. Direct integration gives the constant coefficient $a_0 = 2/\pi$, the single sine coefficient $b_2 = 1/2$, and

$$a_{4n} = -\frac{2}{\pi(2n-1)(2n+1)},$$

with all remaining coefficients equal to zero. This proves the displayed Fourier series.

Parseval gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dxx = \frac{a_0^2}{2} + b_2^2 + \sum_{n=1}^{\infty} a_{4n}^2.$$

The left side is $1/2$, and therefore

$$\frac{1}{2} = \frac{2}{\pi^2} + \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2(2n+1)^2}.$$

Solving gives

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2} = \frac{\pi^2-8}{16}.}$$

Let

$$S(x) = \sum_{n=1}^{\infty} \frac{\sin 4nx}{(2n-1)(2n)(2n+1)}.$$

From the Fourier series,

$$C(x) := \sum_{n=1}^{\infty} \frac{\cos 4nx}{(2n-1)(2n+1)} = \frac{1}{2} + \frac{\pi}{4}\sin 2x - \frac{\pi}{2}f(x).$$

Since $S(x) = 2\int_0^x C(t)\, dxt$, we obtain

$$\boxed{S(x) = \begin{cases} x - \dfrac{\pi}{4}(1-\cos 2x), & 0 \le x \le \frac{\pi}{2}, \\ x - \pi + \dfrac{\pi}{4}(1-\cos 2x), & \frac{\pi}{2} \le x \le \pi. \end{cases}}$$

Exercise 7

Let

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ e^x, & 0 \le x \le \pi, \end{cases}$$

and let F denote the sum of its Fourier series.

(a) Compute a_n, b_n.

(b) Compute $F(0)$, $F(\pi/2)$, and $F(\pi)$.

(c) Describe the graph of F on $[-2\pi, 2\pi]$.

(d) Evaluate $\sum_{n=1}^{\infty}(n^2+1)^{-1}$ and $\sum_{n=1}^{\infty}(-1)^n(n^2+1)^{-1}$.

Solution. Direct integration gives

$$\boxed{a_0 = \frac{e^\pi - 1}{\pi},}$$

$$\boxed{a_n = \frac{(-1)^n e^\pi - 1}{\pi(1+n^2)}, \qquad b_n = \frac{n\big(1-(-1)^n e^\pi\big)}{\pi(1+n^2)}.}$$

At a point of continuity, the series converges to f; at a jump, it converges to the midpoint of the jump. Hence

$$\boxed{F(0) = \frac{1}{2}, \qquad F\Big(\frac{\pi}{2}\Big) = e^{\pi/2}, \qquad F(\pi) = \frac{e^\pi}{2}.}$$

On $[-2\pi, 2\pi]$, F is the 2π-periodic extension:

$$F(x) = \begin{cases} e^{x+2\pi}, & -2\pi < x < -\pi, \\ 0, & -\pi < x < 0, \\ e^x, & 0 < x < \pi, \\ 0, & \pi < x < 2\pi, \end{cases}$$

with midpoint values at the jump points.

Using the values at $x = 0$ and $x = \pi$, or equivalently the standard partial fraction expansion of coth, one obtains

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi\coth\pi - 1}{2},}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi\ \operatorname{csch}\pi - 1}{2}.}$$

Exercise 8

Let

$$f(x) = e^{x(1+2\pi i)}.$$

(a) Find the complex Fourier series on $[-1, 1]$.

(b) Find the real Fourier series on $[-1, 1]$.

(c) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2}.$$

Solution. For period 2, the complex basis is $e^{in\pi x}$. Therefore

$$c_n = \frac{1}{2}\int_{-1}^{1} e^{x(1+2\pi i)} e^{-in\pi x}\,dx x.$$

Set $\alpha_n = 1 + i\pi(2-n)$. Since $\sinh(1+i\pi(2-n)) = (-1)^n \sinh 1$, we get

$$\boxed{c_n = \frac{(-1)^n \sinh 1}{1+i\pi(2-n)}, \qquad n \in \mathbb{Z}.}$$

Thus

$$f(x) \underset{\text{FS}}{\sim} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh 1}{1+i\pi(2-n)} e^{in\pi x}.$$

For the real form

$$f(x) \underset{\text{FS}}{\sim} \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n \cos n\pi x + b_n \sin n\pi x\big),$$

we use $a_0 = 2c_0$, $a_n = c_n + c_{-n}$, and $b_n = i(c_n - c_{-n})$. Hence

$$\boxed{a_0 = \frac{2\sinh 1}{1+2\pi i},}$$

$$\boxed{a_n = (-1)^n \sinh 1 \left[\frac{1}{1+i\pi(2-n)} + \frac{1}{1+i\pi(2+n)}\right],}$$

$$\boxed{b_n = i(-1)^n \sinh 1 \left[\frac{1}{1+i\pi(2-n)} - \frac{1}{1+i\pi(2+n)}\right].}$$

The required sums are the standard identities obtained from the Fourier series of e^x on $[-1,1]$ (or from the above series after shifting indices):

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2} = \frac{\coth 1 - 1}{2},}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2} = \frac{\operatorname{csch} 1 - 1}{2}.}$$

Exercise 9

Let

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases}$$

(a) Find its Fourier series on $[-\pi, \pi]$.

(b) Prove

$$\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots.$$

(c) Prove $\sum_{n=1}^{\infty}(4n^2-1)^{-1} = 1/2$.

Solution. The coefficients are

$$a_0 = \frac{2}{\pi}, \qquad b_1 = \frac{1}{2}, \qquad b_n = 0 \quad (n \neq 1),$$

and

$$a_{2n} = -\frac{2}{\pi(4n^2-1)}, \qquad a_{2n-1} = 0.$$

Therefore

$$\boxed{f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}.}$$

At $x = \pi/2$, $f(\pi/2) = 1$, so

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

Consequently

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)} = \frac{\pi-2}{4}.}$$

At $x = 0$, the Fourier series converges to the midpoint 0, hence

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1},$$

which gives

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.}$$

Exercise 10

On $[-\pi, \pi]$, define

$$f(x) = \begin{cases} \sin 2x, & -\pi/2 \leq x \leq \pi/2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the Fourier series of f.

(b) Find the Fourier series of f'.

(c) To what value does the Fourier series of f' converge at $x = \pm\pi/2$?

(d) Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2(2k+1)^2}, \qquad \sum_{k=1}^{\infty} \frac{(2k-1)(-1)^k}{(2k-3)(2k+1)}.$$

Solution. The function is odd, so only sine coefficients occur. Direct integration yields

$$b_2 = \frac{1}{2}, \qquad b_{2k-1} = \frac{4(-1)^k}{\pi(2k-3)(2k+1)}, \qquad b_{2k} = 0 \quad (k \neq 1).$$

Thus

$$\boxed{f(x) = \frac{1}{2}\sin 2x + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k \sin((2k-1)x)}{(2k-3)(2k+1)}.}$$

Because f is continuous and piecewise C^1, its derivative Fourier series is obtained termwise:

$$\boxed{f'(x) \underset{\text{FS}}{\sim} \cos 2x + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(2k-1)(-1)^k \cos((2k-1)x)}{(2k-3)(2k+1)}.}$$

Classically,

$$f'(x) = \begin{cases} 2\cos 2x, & |x| < \pi/2, \\ 0, & \pi/2 < |x| < \pi. \end{cases}$$

At $x = \pm\pi/2$, the one-sided limits are -2 and 0, so the derivative series converges to

$$\boxed{-1.}$$

Parseval for f gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dxx = \frac{1}{2} = \frac{1}{4} + \frac{16}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2(2k+1)^2}.$$

Therefore

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2(2k+1)^2} = \frac{\pi^2}{64}.}$$

At $x = 0$, the derivative series converges to $f'(0) = 2$, so

$$2 = 1 + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(2k-1)(-1)^k}{(2k-3)(2k+1)}.$$

Hence

$$\boxed{\sum_{k=1}^{\infty} \frac{(2k-1)(-1)^k}{(2k-3)(2k+1)} = \frac{\pi}{4}.}$$

Exercise 11

Let f be piecewise continuous and 2π-periodic, with zero mean and Fourier series

$$f(x) \underset{\text{FS}}{\sim} \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Define

$$g(x) = \int_{-\pi}^{x} \big(f(t) + f(\pi - t)\big)\, dxt,$$

and write

$$g(x) \underset{\text{FS}}{\sim} \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx).$$

Express A_n, B_n in terms of a_n, b_n.

Solution. Using

$$\cos(n(\pi - t)) = (-1)^n \cos nt, \qquad \sin(n(\pi - t)) = (-1)^{n+1} \sin nt,$$

we find

$$f(t) + f(\pi - t) \underset{\text{FS}}{\sim} \sum_{n=1}^{\infty} \left((1 + (-1)^n)a_n \cos nt + (1 - (-1)^n)b_n \sin nt\right).$$

Since $g' = f(t) + f(\pi - t)$, while

$$g'(x) \underset{\text{FS}}{\sim} \sum_{n=1}^{\infty} \left(-nA_n \sin nx + nB_n \cos nx\right),$$

comparison of coefficients gives

$$\boxed{A_n = -\frac{1 - (-1)^n}{n} b_n, \qquad B_n = \frac{1 + (-1)^n}{n} a_n.}$$

Equivalently, $A_n = 0$ for even n and $A_n = -2b_n/n$ for odd n; $B_n = 2a_n/n$ for even n and $B_n = 0$ for odd n.

Exercise 12

Let $C([-\pi,\pi]^n;\mathbb{C})$ be equipped with

$$\langle f,g\rangle = \frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(x_1,\dots,x_n)\overline{g(x_1,\dots,x_n)}\,dx_1\cdots dx_n.$$

(a) Prove that

$$\left\{e^{i(m_1x_1+\cdots+m_nx_n)} : (m_1,\dots,m_n)\in\mathbb{Z}^n\right\}$$

is an orthonormal system.

(b) If

$$f(x_1,\dots,x_n) = f_1(x_1)\cdots f_n(x_n)$$

and $a_m^{(k)}$ is the mth complex Fourier coefficient of f_k, prove

$$c_{m_1,\dots,m_n} = a_{m_1}^{(1)}\cdots a_{m_n}^{(n)}.$$

Solution. For $m,k\in\mathbb{Z}^n$, Fubini's theorem gives

$$\begin{aligned}&\left\langle e^{i(m_1x_1+\cdots+m_nx_n)}, e^{i(k_1x_1+\cdots+k_nx_n)}\right\rangle\\ &= \prod_{j=1}^{n}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m_j-k_j)x_j}\,dx_j\right).\end{aligned}$$

Each factor is 1 if $m_j=k_j$ and 0 otherwise. Thus the product equals 1 when $m=k$ and 0 otherwise, proving orthonormality.

For the product function, again by Fubini,

$$\begin{aligned}c_{m_1,\dots,m_n} &= \frac{1}{(2\pi)^n}\int\cdots\int\prod_{j=1}^{n} f_j(x_j)e^{-im_jx_j}\,dx_1\cdots dx_n\\ &= \prod_{j=1}^{n}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f_j(x_j)e^{-im_jx_j}\,dx_j\right)\\ &= \boxed{a_{m_1}^{(1)}a_{m_2}^{(2)}\cdots a_{m_n}^{(n)}}.\end{aligned}$$

Editorial notes

- In Exercise 11, the original Fourier series of f begins at $n=1$, so the problem implicitly assumes that f has mean zero. This is needed for the integral defining g to be 2π-periodic.
- In Exercise 12, the complex inner product must include a conjugate on the second function. The scanned text appears to lose that overline typographically; it has been restored here.
- All identities at discontinuities use the midpoint convention supplied by the Dirichlet convergence theorem.

Chapter 3, Section 2
Fourier Transform Exercises

Convention

Throughout we use

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx.$$

At removable singularities, the displayed formulas are interpreted by continuity.

Exercise 1

Compute the Fourier transform of each function.

(a) The triangular function

Let

$$f_a(x) = \begin{cases} 1 - \dfrac{|x|}{a}, & |x| < a, \\ 0, & |x| \geq a, \end{cases} \qquad a > 0.$$

The function is even, hence

$$\widehat{f}_a(\xi) = 2\int_0^a \left(1 - \frac{x}{a}\right)\cos(\xi x)\,dx.$$

A direct integration gives

$$\widehat{f}_a(\xi) = \frac{2(1-\cos(a\xi))}{a\xi^2} = a\left(\frac{\sin(a\xi/2)}{a\xi/2}\right)^2.$$

Thus

$$\boxed{\widehat{f}_a(\xi) = a\,\mathrm{sinc}^2\left(\frac{a\xi}{2}\right)}.$$

In particular, $\widehat{f}_a(0) = a$, equal to the area of the triangle.

(b) The indicator of $[0,1]$

Let

$$f(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\widehat{f}(\xi) = \int_0^1 e^{-i\xi x}\,dx = \frac{1-e^{-i\xi}}{i\xi} = e^{-i\xi/2}\frac{2\sin(\xi/2)}{\xi}.$$

Therefore

$$\boxed{\widehat{f}(\xi) = \frac{1-e^{-i\xi}}{i\xi}}.$$

At $\xi = 0$, the value is 1.

(c) A two-level even pulse

Let

$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 2, & 1 < |x| < 2, \\ 0, & |x| \ge 2. \end{cases}$$

Since f is even,

$$\begin{aligned} \widehat{f}(\xi) &= 2\int_0^1 \cos(\xi x)\,dx + 4\int_1^2 \cos(\xi x)\,dx \\ &= \frac{2\sin\xi}{\xi} + \frac{4(\sin 2\xi - \sin\xi)}{\xi}. \end{aligned}$$

Hence

$$\boxed{\widehat{f}(\xi) = \frac{4\sin(2\xi) - 2\sin\xi}{\xi}}.$$

The continuous value at $\xi = 0$ is 6, the total area of f.

(d) A truncated sine

Let

$$f(x) = \begin{cases} \sin x, & |x| \le \pi, \\ 0, & |x| > \pi. \end{cases}$$

This function is odd, so

$$\widehat{f}(\xi) = -2i\int_0^\pi \sin x\,\sin(\xi x)\,dx.$$

Using the product-to-sum identity,

$$\int_0^\pi \sin x \sin(\xi x)\,dx = \frac{\sin(\pi\xi)}{1-\xi^2}.$$

Thus

$$\boxed{\widehat{f}(\xi) = \frac{2i\sin(\pi\xi)}{\xi^2 - 1}}.$$

At the removable singularities,

$$\widehat{f}(1) = -i\pi, \qquad \widehat{f}(-1) = i\pi.$$

(e) A truncated linear function

Let

$$f(x) = \begin{cases} x, & |x| \le a, \\ 0, & |x| > a, \end{cases} \qquad a > 0.$$

The function is odd, hence

$$\widehat{f}(\xi) = -2i\int_0^a x\sin(\xi x)\,dx.$$

Integration by parts yields

$$\int_0^a x\sin(\xi x)\,dx = -\frac{a\cos(a\xi)}{\xi} + \frac{\sin(a\xi)}{\xi^2}.$$

Therefore

$$\boxed{\widehat{f}(\xi) = 2i\left(\frac{a\cos(a\xi)}{\xi} - \frac{\sin(a\xi)}{\xi^2}\right) = \frac{2i\big(a\xi\cos(a\xi) - \sin(a\xi)\big)}{\xi^2}}.$$

The continuous value at $\xi = 0$ is 0, as expected for an odd integrable function.

(f) A truncated quadratic

Let

$$f(x) = \begin{cases} x^2, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

Since f is even,

$$\widehat{f}(\xi) = 2\int_0^1 x^2 \cos(\xi x)\, dx.$$

Twice integrating by parts gives

$$\boxed{\widehat{f}(\xi) = 2\left(\frac{\sin\xi}{\xi} + \frac{2\cos\xi}{\xi^2} - \frac{2\sin\xi}{\xi^3}\right) = \frac{2\big((\xi^2-2)\sin\xi + 2\xi\cos\xi\big)}{\xi^3}}.$$

At $\xi = 0$, $\widehat{f}(0) = \int_{-1}^1 x^2 dx = 2/3$.

(g) A truncated cosine

Let

$$f(x) = \begin{cases} \cos x, & |x| \le \pi, \\ 0, & |x| > \pi. \end{cases}$$

The function is even, and therefore

$$\widehat{f}(\xi) = 2\int_0^\pi \cos x \cos(\xi x)\, dx.$$

Using product-to-sum,

$$\boxed{\widehat{f}(\xi) = \frac{2\xi\sin(\pi\xi)}{1-\xi^2}}.$$

The removable values are

$$\widehat{f}(1) = \widehat{f}(-1) = \pi.$$

(h) An odd exponential function

Let

$$f(x) = \begin{cases} e^x, & x < 0, \\ -e^{-x}, & x > 0. \end{cases}$$

Then

$$\begin{aligned} \widehat{f}(\xi) &= \int_{-\infty}^0 e^{(1-i\xi)x}\, dx - \int_0^\infty e^{-(1+i\xi)x}\, dx \\ &= \frac{1}{1-i\xi} - \frac{1}{1+i\xi}. \end{aligned}$$

Hence

$$\boxed{\widehat{f}(\xi) = \frac{2i\xi}{1+\xi^2}}.$$

(i) The function $f(x) = |x|e^{-|x|}$

The function is even, so

$$\widehat{f}(\xi) = 2\int_0^\infty xe^{-x}\cos(\xi x)\, dx.$$

Since

$$\int_0^\infty xe^{-(1-i\xi)x}\,dx = \frac{1}{(1-i\xi)^2},$$

we take the real part and obtain

$$\boxed{\widehat{f}(\xi) = \frac{2(1-\xi^2)}{(1+\xi^2)^2}}.$$

At $\xi = 0$, this gives 2, equal to $\int_{-\infty}^{\infty} |x|e^{-|x|}dx$.

Exercise 2: The Gaussian integral

Prove that

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

The integrand is positive, so Tonelli's theorem permits us to write

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\ &= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy. \end{aligned}$$

Pass to polar coordinates,

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dx\,dy = r\,dr\,d\theta.$$

Then

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta \\ &= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \\ &= 2\pi\cdot\frac{1}{2} = \pi. \end{aligned}$$

Since $I > 0$,

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}.$$

Editorial note

The Hebrew source does not state the Fourier-transform normalization on this page. The solutions above use the common convention $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx$. If the book uses a normalized convention, the answers differ only by the corresponding global normalization and/or frequency scaling.

Chapter 3, Section 3
Fourier Transform Properties

Throughout, we use the convention

$$\widehat{f}(\omega) = \mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}\,d\omega.$$

The basic operational rules used below are

$$\mathcal{F}[f'](\omega) = i\omega\widehat{f}(\omega), \qquad \mathcal{F}[xf(x)](\omega) = i\widehat{f}\,'(\omega), \qquad \mathcal{F}[x^2 f(x)](\omega) = -\widehat{f}\,''(\omega).$$

Exercise 1

Compute the Fourier transform of each function.

(a) $f(x) = e^{-a|x|}$, $a > 0$

Since f is even,

$$\begin{aligned}\widehat{f}(\omega) &= 2\int_0^{\infty} e^{-ax}\cos(\omega x)\,dx \\ &= 2\,\frac{a}{a^2+\omega^2}.\end{aligned}$$

Thus

$$\boxed{\widehat{f}(\omega) = \frac{2a}{a^2+\omega^2}.}$$

(b) $f(x) = e^{-4x^2-4x-1}$

Complete the square:

$$-4x^2 - 4x - 1 = -4\left(x + \frac{1}{2}\right)^2.$$

Let $g(x) = e^{-4x^2}$. Then $f(x) = g(x + \frac{1}{2})$, and

$$\widehat{g}(\omega) = \sqrt{\frac{\pi}{4}}e^{-\omega^2/16} = \frac{\sqrt{\pi}}{2}e^{-\omega^2/16}.$$

The translation rule gives

$$\boxed{\widehat{f}(\omega) = e^{i\omega/2}\frac{\sqrt{\pi}}{2}e^{-\omega^2/16}.}$$

(c) $f(x) = 4xe^{-x^2}$

We know

$$\mathcal{F}[e^{-x^2}](\omega) = \sqrt{\pi}e^{-\omega^2/4}.$$

Therefore

$$\begin{aligned}\widehat{f}(\omega) &= 4i\frac{d}{d\omega}\left(\sqrt{\pi}e^{-\omega^2/4}\right) \\ &= -2i\sqrt{\pi}\,\omega e^{-\omega^2/4}.\end{aligned}$$

Hence

$$\boxed{\widehat{f}(\omega) = -2i\sqrt{\pi}\,\omega e^{-\omega^2/4}.}$$

Exercise 2

Let

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Find the Fourier transforms below.

(a) $f(x) = H(x)e^{-ax}$, $a > 0$

$$\widehat{f}(\omega) = \int_0^\infty e^{-(a+i\omega)x}\,dx = \boxed{\frac{1}{a+i\omega}}.$$

(b) $f(x) = H(x)e^{-ax}\cos bx$, $a > 0$, $b \neq 0$

Using $\cos bx = (e^{ibx} + e^{-ibx})/2$,

$$\begin{aligned}\widehat{f}(\omega) &= \frac{1}{2}\left[\frac{1}{a+i(\omega-b)} + \frac{1}{a+i(\omega+b)}\right] \\ &= \boxed{\frac{a+i\omega}{(a+i\omega)^2+b^2}}.\end{aligned}$$

(c) $g(x) = H(x)e^{-ax}\sin bx$, $a > 0$, $b \neq 0$

Similarly,

$$\begin{aligned}\widehat{g}(\omega) &= \frac{1}{2i}\left[\frac{1}{a+i(\omega-b)} - \frac{1}{a+i(\omega+b)}\right] \\ &= \boxed{\frac{b}{(a+i\omega)^2+b^2}}.\end{aligned}$$

Exercise 3

Let $f \in G(\mathbb{R})$ and write $F(\omega) = \widehat{f}(\omega)$. Compute the transforms of the following functions.

(a) Reflection:

$$\boxed{\mathcal{F}[f(-x)](\omega) = F(-\omega).}$$

(b) Translation by $x_0 \in \mathbb{R}$:

$$\boxed{\mathcal{F}[f(x-x_0)](\omega) = e^{-i\omega x_0}F(\omega).}$$

(c) Modulation by $\omega_0 \in \mathbb{R}$:

$$\boxed{\mathcal{F}[e^{i\omega_0 x}f(x)](\omega) = F(\omega-\omega_0).}$$

(d) Multiplication by $\sin(\omega_0 x)$:

$$\boxed{\mathcal{F}[f(x)\sin(\omega_0 x)](\omega) = \frac{F(\omega-\omega_0)-F(\omega+\omega_0)}{2i}.}$$

(e) Multiplication by $\cos(\omega_0 x)$:

$$\boxed{\mathcal{F}[f(x)\cos(\omega_0 x)](\omega) = \frac{F(\omega-\omega_0)+F(\omega+\omega_0)}{2}.}$$

(f) $e^{ix}f(3x)$: First, $\mathcal{F}[f(3x)](\omega) = \frac{1}{3}F(\omega/3)$. Modulation by e^{ix} then gives

$$\boxed{\mathcal{F}[e^{ix}f(3x)](\omega) = \frac{1}{3}F\left(\frac{\omega-1}{3}\right).}$$

(g) $f(2x)$:

$$\boxed{\mathcal{F}[f(2x)](\omega) = \frac{1}{2}F\left(\frac{\omega}{2}\right).}$$

Exercise 4

Suppose y has a continuous second derivative and is bounded and absolutely integrable on $\mathbb{R}$, and

$$y''(x) + 2xy'(x) + 2y(x) = 0.$$

Let $Y(\omega) = \mathcal{F}[y](\omega)$. We use

$$\mathcal{F}[y''] = -\omega^2 Y, \qquad \mathcal{F}[xy'] = i\frac{d}{d\omega}(i\omega Y) = -(Y+\omega Y').$$

Transforming the differential equation gives

$$-\omega^2 Y - 2(Y+\omega Y') + 2Y = 0,$$

so

$$\omega^2 Y + 2\omega Y' = 0.$$

Thus the transformed function satisfies

$$\boxed{2Y'(\omega) + \omega Y(\omega) = 0}$$

(for $\omega \neq 0$, and by continuity also in the natural extended sense at $\omega = 0$).

Exercise 5

Use the Fourier transform to solve

$$y'' + xy' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Let $Y = \mathcal{F}[y]$. Since

$$\mathcal{F}[xy'] = -(Y+\omega Y'),$$

we obtain

$$-\omega^2 Y - (Y + \omega Y') + Y = 0,$$

that is,

$$\omega Y' + \omega^2 Y = 0.$$

For $\omega \neq 0$,

$$Y' + \omega Y = 0,$$

and hence

$$Y(\omega) = Ce^{-\omega^2/2}.$$

Using the inverse transform at $x = 0$,

$$1 = y(0) = \frac{C}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2/2}\, d\omega = \frac{C}{\sqrt{2\pi}},$$

so $C = \sqrt{2\pi}$. Since

$$\mathcal{F}[e^{-x^2/2}](\omega) = \sqrt{2\pi} e^{-\omega^2/2},$$

we conclude that

$$\boxed{y(x) = e^{-x^2/2}.}$$

Indeed, $y(0) = 1$ and $y'(0) = 0$.

Exercise 6

Assume f, f', f'', tf, and $t^2 f$ are continuous and absolutely integrable on $\mathbb{R}$. Let $F = \mathcal{F}[f]$. Find $c \in \mathbb{R}$ such that

$$f''(t) + (t^2 - 2)f(t) = cf(t)$$

implies

$$F''(\omega) + (\omega^2 - 2)F(\omega) = cF(\omega).$$

Transforming the first equation,

$$-\omega^2 F - F'' - 2F = cF.$$

Equivalently,

$$F'' + (\omega^2 + 2)F = -cF.$$

For this to equal

$$F'' + (\omega^2 - 2)F = cF,$$

the constant terms must agree:

$$2 + c = -2 - c.$$

Therefore

$$\boxed{c = -2.}$$

For this value, both transformed equations reduce to

$$F''(\omega) + \omega^2 F(\omega) = 0.$$

Exercise 7

Given

$$F(\omega) = \mathcal{F}[f](\omega) = \frac{1}{1 + \omega^2},$$

compute

$$\mathcal{F}\big[x^2 f''(x) + 2f'''(x)\big](\omega).$$

First,

$$\mathcal{F}[f''] = -\omega^2 F,$$

so

$$\mathcal{F}[x^2 f''] = -\frac{d^2}{d\omega^2}(-\omega^2 F) = \frac{d^2}{d\omega^2}(\omega^2 F).$$

Also,

$$\mathcal{F}[2f'''] = 2(i\omega)^3 F = -2i\omega^3 F.$$

Hence

$$\mathcal{F}[x^2 f'' + 2f'''] = \left(\frac{\omega^2}{1+\omega^2}\right)'' - \frac{2i\omega^3}{1+\omega^2}.$$

Since

$$\left(\frac{\omega^2}{1+\omega^2}\right)'' = \frac{2(1-3\omega^2)}{(1+\omega^2)^3},$$

we obtain

$$\boxed{\mathcal{F}[x^2 f'' + 2f'''](\omega) = \frac{2(1-3\omega^2)}{(1+\omega^2)^3} - \frac{2i\omega^3}{1+\omega^2}.}$$

Editorial note

The source uses the notation $\mathcal{F}[y]$ for the Fourier transform and assumes sufficient decay and differentiability to justify differentiation under the integral sign and integration by parts. The solutions above retain those assumptions and the same transform normalization used in Chapter 3, Section 1.

Chapter 3, Section 4
Fourier Transform Exercises

Convention. Throughout,

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}\,d\omega.$$

Parseval's identity is

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\,d\omega.$$

Exercise 1

Assume $a > 0$. A standard transform pair is

$$\mathcal{F}\left\{\frac{1}{x^2+a^2}\right\}(\omega) = \frac{\pi}{a}e^{-a|\omega|}.$$

(a) Therefore

$$\boxed{\widehat{f}(\omega) = \frac{\pi}{a}e^{-a|\omega|}} \qquad \left(f(x) = \frac{1}{x^2+a^2}\right).$$

(b) Using $\cos(bx) = \frac{1}{2}(e^{ibx}+e^{-ibx})$ and the modulation rule,

$$\mathcal{F}\left\{\frac{\cos(bx)}{x^2+a^2}\right\}(\omega) = \frac{1}{2}\left[\frac{\pi}{a}e^{-a|\omega-b|} + \frac{\pi}{a}e^{-a|\omega+b|}\right].$$

Hence

$$\boxed{\widehat{f}(\omega) = \frac{\pi}{2a}\left(e^{-a|\omega-b|} + e^{-a|\omega+b|}\right)}.$$

In the printed exercise the cosine parameter is also denoted by a; the formula is obtained by setting $b = a$.

(c) Similarly, using $\sin(bx) = \frac{1}{2i}(e^{ibx}-e^{-ibx})$,

$$\boxed{\mathcal{F}\left\{\frac{\sin(bx)}{x^2+a^2}\right\}(\omega) = \frac{\pi}{2ai}\left(e^{-a|\omega-b|} - e^{-a|\omega+b|}\right)}.$$

Exercise 2

We use earlier transform pairs and Parseval's identity.

(a) For $\chi_{[-1,1]}$, the transform is $2\sin\omega/\omega$. Thus

$$2 = \int_{-1}^{1} 1\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{2\sin\omega}{\omega}\right|^2 d\omega.$$

By evenness,

$$\boxed{\int_0^{\infty}\frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}}.$$

(b) The triangular function $g(x) = (1-|x|)_+$ has transform

$$\widehat{g}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2.$$

Since $\int |g|^2 = 2/3$, Parseval gives

$$\boxed{\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}}.$$

(c) The transform of the unit ball in $\mathbb{R}^3$, after radial reduction, contains $\big(\sin u - u\cos u\big)/u^3$. The corresponding Parseval identity yields

$$\boxed{\int_0^\infty \left(\frac{\sin u - u\cos u}{u^2}\right)^2 du = \frac{\pi}{6}}.$$

(d) Since

$$\frac{u^2}{(1+u^2)^2} = \frac{1}{1+u^2} - \frac{1}{(1+u^2)^2},$$

and

$$\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{2}, \qquad \int_0^\infty \frac{du}{(1+u^2)^2} = \frac{\pi}{4},$$

we obtain

$$\boxed{\int_0^\infty \frac{u^2}{(1+u^2)^2}\,du = \frac{\pi}{4}}.$$

(e) Directly,

$$\boxed{\int_0^\infty \frac{du}{(1+u^2)^2} = \frac{\pi}{4}}.$$

(f) From the transform pair associated with $\cos(\pi x)\chi_{[-1,1]}(x)$, or by contour integration after removal of the singularity at $x=1$,

$$\boxed{\int_0^\infty \frac{x\sin(\pi x)}{1-x^2}\,dx = \frac{\pi}{2}}.$$

The integrand is interpreted by continuity at $x=1$.

(g) Using $\sin(\pi x)\cos(\pi x) = \frac{1}{2}\sin(2\pi x)$,

$$\boxed{\int_0^\infty \frac{x\sin(\pi x)\cos(\pi x)}{1-x^2}\,dx = -\frac{\pi}{4}}.$$

(h) Parseval for the same transform pair gives

$$\boxed{\int_0^\infty \left(\frac{x\sin(\pi x)}{1-x^2}\right)^2 dx = \frac{\pi^2}{4}}.$$

Exercise 3

Let $a>0$ and $b>0$. Use

$$\cos(ax) - 1 = -2\sin^2\frac{ax}{2}$$

or, more directly,

$$(\cos ax - 1)\sin bx = \frac{1}{2}\big[\sin((b+a)x) + \sin((b-a)x) - 2\sin(bx)\big].$$

The Dirichlet integral

$$\int_0^\infty \frac{\sin(cx)}{x}\,dx = \frac{\pi}{2}\,\mathrm{sgn}(c) \qquad (c \neq 0)$$

therefore gives

$$\boxed{\int_0^\infty \frac{\cos(ax)-1}{x}\sin(bx)\,dx = \begin{cases} -\frac{\pi}{2}, & 0<b<a, \\ -\frac{\pi}{4}, & b=a, \\ 0, & b>a. \end{cases}}$$

Exercise 4

For $a, b > 0$, $a \neq b$,

$$\frac{1}{(t^2+a^2)(t^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{t^2+a^2} - \frac{1}{t^2+b^2}\right).$$

Thus

$$\int_0^\infty \frac{dt}{(t^2+a^2)(t^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{\pi}{2a} - \frac{\pi}{2b}\right) = \boxed{\frac{\pi}{2ab(a+b)}}.$$

The same formula follows at $a = b$ by continuity.

Exercise 5

We are given

$$F(\omega) = \begin{cases} 1-\omega^2, & |\omega| \le 1, \\ 0, & |\omega| > 1. \end{cases}$$

By Fourier inversion,

$$f(x) = \frac{1}{2\pi}\int_{-1}^{1} (1-\omega^2)e^{i\omega x}\,d\omega = \frac{1}{\pi}\int_0^1 (1-\omega^2)\cos(\omega x)\,d\omega.$$

A direct integration gives, for $x \neq 0$,

$$\int_0^1 (1-\omega^2)\cos(\omega x)\,d\omega = \frac{2(\sin x - x\cos x)}{x^3}.$$

Hence

$$\boxed{f(x) = \frac{2}{\pi}\frac{\sin x - x\cos x}{x^3}, \qquad x \neq 0.}$$

At $x = 0$ the continuous value is

$$\boxed{f(0) = \frac{2}{3\pi}}.$$

Exercise 6

The printed assumptions are inconsistent. It states that $f : \mathbb{R} \to \mathbb{C}$ is odd, continuous, and absolutely integrable, while

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \begin{cases} 1-\omega, & |\omega| \le 1, \\ 0, & |\omega| > 1. \end{cases}$$

If f is odd, then its Fourier transform must be odd:

$$\widehat{f}(-\omega) = -\widehat{f}(\omega).$$

But the displayed right-hand side is not odd. Therefore no function can satisfy all the printed assumptions simultaneously.

If one ignores the word “odd”, then inversion gives

$$f(x) = \int_{-1}^{1} (1 - \omega) e^{i\omega x} \, d\omega,$$

which can be evaluated explicitly, but the resulting function is not odd. Thus the source appears to contain a typographical error in Exercise 6.

Chapter 3, Section 5
Fourier Transform: Convolution and Integral Equations

Fourier-transform convention

Throughout,

$$\widehat{f}(\omega) = \mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}\,d\omega.$$

For convolution,

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt, \qquad \widehat{f * g} = \widehat{f}\,\widehat{g}.$$

We repeatedly use

$$\mathcal{F}[e^{-a|x|}](\omega) = \frac{2a}{a^2+\omega^2}, \qquad \mathcal{F}\left[\frac{1}{x^2+a^2}\right](\omega) = \frac{\pi}{a}e^{-a|\omega|}, \quad a>0,$$

and

$$\mathcal{F}[e^{-\alpha x^2}](\omega) = \sqrt{\frac{\pi}{\alpha}}\,e^{-\omega^2/(4\alpha)}, \qquad \alpha > 0.$$

Exercise 1

Use the Fourier transform to solve the following integral equations.

(a)

$$\int_{-\infty}^{\infty} f(t)f(x-t)\,dt = e^{-x^2/2}.$$

Solution. The equation is

$$f * f = e^{-x^2/2}.$$

Taking Fourier transforms gives

$$\widehat{f}(\omega)^2 = \mathcal{F}[e^{-x^2/2}](\omega) = \sqrt{2\pi}\,e^{-\omega^2/2}.$$

If $\widehat{f}$ is continuous, then the right-hand side never vanishes, so the sign of the square root is constant. Hence

$$\widehat{f}(\omega) = \pm(2\pi)^{1/4}e^{-\omega^2/4}.$$

Since

$$\mathcal{F}[e^{-x^2}](\omega) = \sqrt{\pi}\,e^{-\omega^2/4},$$

we obtain

$$\boxed{f(x) = \pm\left(\frac{2}{\pi}\right)^{1/4} e^{-x^2}.}$$

A direct convolution check confirms both signs, because the sign disappears in $f * f$.

(b)

For $0 < a < b$, solve

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + a^2}\, dt = \frac{1}{x^2 + b^2}.$$

Solution. This is

$$f * k_a = k_b, \qquad k_c(x) = \frac{1}{x^2 + c^2}.$$

Taking transforms,

$$\widehat{f}(\omega)\frac{\pi}{a}e^{-a|\omega|} = \frac{\pi}{b}e^{-b|\omega|}.$$

Therefore

$$\widehat{f}(\omega) = \frac{a}{b}e^{-(b-a)|\omega|}.$$

For $c > 0$,

$$\mathcal{F}^{-1}[e^{-c|\omega|}](x) = \frac{c}{\pi(x^2 + c^2)}.$$

With $c = b - a$,

$$\boxed{f(x) = \frac{a(b-a)}{b\pi\big(x^2 + (b-a)^2\big)}.}$$

(c)

$$\int_{-\infty}^{\infty} f(t)f(x-t)\, dt = \frac{1}{x^2+1}.$$

Solution. Taking Fourier transforms gives

$$\widehat{f}(\omega)^2 = \pi e^{-|\omega|}.$$

Under the same continuity assumption as in part (a),

$$\widehat{f}(\omega) = \pm\sqrt{\pi}\, e^{-|\omega|/2}.$$

Using

$$\mathcal{F}^{-1}[e^{-c|\omega|}](x) = \frac{c}{\pi(x^2 + c^2)},$$

with $c = 1/2$, we get

$$\boxed{f(x) = \pm\frac{1}{2\sqrt{\pi}\,(x^2 + 1/4)} = \pm\frac{2}{\sqrt{\pi}\,(4x^2+1)}.}$$

(d)

$$\int_{-\infty}^{\infty} e^{-|x-t|} f(t)\, dt = e^{-|x|} + |x|e^{-|x|}.$$

Solution. The left side is $e^{-|\cdot|} * f$. We have

$$\mathcal{F}[e^{-|x|}](\omega) = \frac{2}{1+\omega^2}.$$

Also, differentiating $\mathcal{F}[e^{-a|x|}] = 2a/(a^2+\omega^2)$ with respect to a,

$$\mathcal{F}[|x|e^{-|x|}](\omega) = \frac{2(1-\omega^2)}{(1+\omega^2)^2}.$$

Hence the transform of the right side is

$$\frac{2}{1+\omega^2} + \frac{2(1-\omega^2)}{(1+\omega^2)^2} = \frac{4}{(1+\omega^2)^2}.$$

Therefore

$$\frac{2}{1+\omega^2}\widehat{f}(\omega) = \frac{4}{(1+\omega^2)^2},$$

so

$$\widehat{f}(\omega) = \frac{2}{1+\omega^2}.$$

Thus

$$\boxed{f(x) = e^{-|x|}.}$$

Exercise 2

For $a > 0$, define

$$f_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| \geq a, \end{cases} \qquad g_a(x) = \begin{cases} 1 - |x|/a, & |x| < a, \\ 0, & |x| \geq a. \end{cases}$$

Compute $f_a * f_a$ and $g_a * g_a$.

Solution. For $f_a * f_a$, the integrand is 1 precisely where the intervals $(-a, a)$ and $(x-a, x+a)$ overlap. The convolution is therefore the length of that overlap:

$$\boxed{(f_a * f_a)(x) = \begin{cases} 2a - |x|, & |x| < 2a, \\ 0, & |x| \geq 2a. \end{cases}}$$

For $g_a * g_a$, write

$$g_a(x) = T(x/a), \qquad T(u) = (1 - |u|)_+.$$

Then, after the substitution $t = au$,

$$(g_a * g_a)(x) = a\,(T * T)(x/a).$$

A direct piecewise integration gives, with $s = |x|/a$,

$$(T * T)(s) = \begin{cases} \dfrac{2}{3} - s^2 + \dfrac{s^3}{2}, & 0 \leq s \leq 1, \\ \dfrac{(2-s)^3}{6}, & 1 \leq s \leq 2, \\ 0, & s \geq 2. \end{cases}$$

Thus

$$\boxed{(g_a * g_a)(x) = \begin{cases} \dfrac{2a}{3} - \dfrac{x^2}{a} + \dfrac{|x|^3}{2a^2}, & |x| \leq a, \\ \dfrac{(2a - |x|)^3}{6a^2}, & a \leq |x| \leq 2a, \\ 0, & |x| \geq 2a. \end{cases}}$$

The two formulas agree at $|x| = a$, where both equal $a/6$.

Exercise 3

Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and absolutely integrable, and let $F = \widehat{f}$. Suppose

$$F(\omega) + \int_{-\infty}^{\infty} F(\omega - s)e^{-|s|}\,ds = \begin{cases} \omega^2, & 0 \le \omega \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find f.

Solution. As printed, this problem has no solution in the stated class.

Indeed, because $f \in L^1(\mathbb{R})$, its Fourier transform F is continuous. The convolution

$$(F * e^{-|\cdot|})(\omega) = \int_{-\infty}^{\infty} F(\omega - s)e^{-|s|}\,ds$$

is also continuous. Therefore the entire left-hand side is continuous in ω.

However, the right-hand side has a jump at $\omega = 1$: its left-hand value is 1, whereas its right-hand value is 0. This is impossible.

Consequently,

$$\boxed{\text{there is no continuous absolutely integrable } f \text{ satisfying the equation as printed.}}$$

This strongly suggests a typographical error in the right-hand side of the original exercise.

Exercise 4

Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and absolutely integrable, and let $F = \widehat{f}$. Assume

$$F(\omega) = 0 \qquad \text{for } |\omega| > \omega_0.$$

Show that for every $a > \omega_0$,

$$f * \frac{\sin(ax)}{\pi x} = f.$$

Solution. Let

$$k_a(x) = \frac{\sin(ax)}{\pi x},$$

with the continuous value $k_a(0) = a/\pi$. The standard transform pair is

$$\widehat{k_a}(\omega) = \mathbf{1}_{[-a,a]}(\omega),$$

up to irrelevant endpoint values.

Therefore

$$\mathcal{F}[f * k_a](\omega) = F(\omega)\mathbf{1}_{[-a,a]}(\omega).$$

Since $a > \omega_0$ and $F(\omega) = 0$ outside $[-\omega_0, \omega_0]$,

$$F(\omega)\mathbf{1}_{[-a,a]}(\omega) = F(\omega).$$

Taking inverse transforms gives

$$\boxed{f(x) * \frac{\sin(ax)}{\pi x} = f(x).}$$

Thus convolution with the sinc kernel reproduces every signal whose frequency support lies inside $[-a, a]$.

Exercise 5

Let

$$f(x) = \begin{cases} 1, & 0 \le x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and let $F = \widehat{f}$. Find a function g whose Fourier transform G satisfies

$$G(\omega) = [F(\omega)]^2.$$

Solution. By the convolution theorem,

$$\widehat{f * f}(\omega) = F(\omega)^2.$$

Hence we may take

$$g = f * f.$$

Now $(f * f)(x)$ equals the length of the intersection of $[0, 1]$ and $[x - 1, x]$. Therefore

$$\boxed{g(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}}$$

This triangular function has Fourier transform $G = F^2$, as required.

Editorial notes

- In Exercise 3, the right-hand side is discontinuous at $\omega = 1$, while the left-hand side must be continuous under the stated hypotheses. Thus the exercise, exactly as printed, has no solution.
- In Exercise 5, the right-to-left typesetting can make the formula look ambiguous. The intended relation is $G(\omega) = [F(\omega)]^2$, which leads directly to $g = f * f$.

Chapter 3 – Summary Exercises
Solutions to Six Fourier Transform Exercises

Fourier transform convention

Throughout the solutions we use

$$\mathcal{F}[f](\omega) = \widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx, \qquad f(x) = \int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}\,d\omega.$$

With this normalization,

$$\mathcal{F}[f*g] = 2\pi\widehat{f}\,\widehat{g}, \qquad \int_{\mathbb{R}} |f(x)|^2\,dx = 2\pi\int_{\mathbb{R}} |\widehat{f}(\omega)|^2\,d\omega.$$

Exercise 1

Let

$$f(x) = \begin{cases} 1-x^2, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

Compute its Fourier transform and use it to evaluate the stated cosine integral.

Solution

Since f is even,

$$\begin{aligned} \widehat{f}(\omega) &= \frac{1}{\pi}\int_0^1 (1-x^2)\cos(\omega x)\,dx \\ &= \frac{2}{\pi}\,\frac{\sin\omega - \omega\cos\omega}{\omega^3}, \qquad \omega \neq 0. \end{aligned}$$

At $\omega = 0$ the removable singularity is filled by

$$\widehat{f}(0) = \frac{1}{2\pi}\int_{-1}^{1} (1-x^2)\,dx = \frac{2}{3\pi}.$$

Thus

$$\boxed{\widehat{f}(\omega) = \frac{2(\sin\omega - \omega\cos\omega)}{\pi\omega^3}}$$

with the value at $\omega = 0$ understood by continuity.

Because f and $\widehat{f}$ are even, Fourier inversion gives

$$f(y) = 2\int_0^{\infty} \widehat{f}(\omega)\cos(\omega y)\,d\omega.$$

Taking $y = \frac{1}{2}$ yields

$$\int_0^{\infty} \widehat{f}(\omega)\cos\frac{\omega}{2}\,d\omega = \frac{1}{2}f\left(\frac{1}{2}\right) = \frac{3}{8}.$$

Therefore

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{\pi}{2}\int_0^\infty \widehat{f}(x)\cos\frac{x}{2}\,dx$$
$$= \boxed{-\frac{3\pi}{16}}.$$

Remark. The source sheet prints $+3\pi/16$. With the numerator as printed, $x\cos x - \sin x$, the correct sign is negative. The value would be $+3\pi/16$ if the numerator were $\sin x - x\cos x$.

Exercise 2

For $x > 0$, let $f(x) = e^{-x}\cos x$, and let $\widetilde{f}$ be its odd extension to $\mathbb{R}$. Prove that, for $x \neq 0$,

$$\widetilde{f}(x) = \frac{2}{\pi}\int_0^\infty \frac{t^3\sin(xt)}{t^4+4}\,dt.$$

Solution

The Fourier sine transform of f is

$$\begin{aligned} S_f(t) &= \int_0^\infty e^{-x}\cos x\sin(tx)\,dx \\ &= \frac{1}{2}\int_0^\infty e^{-x}\big(\sin((t+1)x) + \sin((t-1)x)\big)\,dx \\ &= \frac{1}{2}\left(\frac{t+1}{1+(t+1)^2} + \frac{t-1}{1+(t-1)^2}\right) \\ &= \frac{t^3}{t^4+4}. \end{aligned}$$

The sine inversion formula gives, for $x > 0$,

$$f(x) = \frac{2}{\pi}\int_0^\infty S_f(t)\sin(xt)\,dt = \frac{2}{\pi}\int_0^\infty \frac{t^3\sin(xt)}{t^4+4}\,dt.$$

Both sides are odd functions of x, so the same identity holds for every $x \neq 0$:

$$\boxed{\widetilde{f}(x) = \frac{2}{\pi}\int_0^\infty \frac{t^3\sin(xt)}{t^4+4}\,dt}.$$

Exercise 3

Let

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

(a) Fourier transform

We have

$$\widehat{f}(\omega) = \frac{1}{2\pi}\int_0^\infty e^{-(1+i\omega)x}\,dx = \boxed{\frac{1}{2\pi(1+i\omega)}}.$$

(b) The convolutions

For $x > 0$,

$$\begin{aligned}(f * f)(x) &= \int_{-\infty}^{\infty} f(t)f(x-t)\,dt \\ &= \int_0^x e^{-t}e^{-(x-t)}\,dt = xe^{-x}.\end{aligned}$$

For $x \le 0$ the convolution is zero. Hence

$$\boxed{(f * f)(x) = xe^{-x}\mathbf{1}_{(0,\infty)}(x)}.$$

Convolving this function with itself, for $x > 0$,

$$\begin{aligned}((f * f) * (f * f))(x) &= \int_0^x te^{-t}(x-t)e^{-(x-t)}\,dt \\ &= e^{-x}\int_0^x t(x-t)\,dt = \frac{x^3}{6}e^{-x}.\end{aligned}$$

Thus

$$\boxed{((f * f) * (f * f))(x) = \frac{x^3}{6}e^{-x}\mathbf{1}_{(0,\infty)}(x)}.$$

(c) Fourier transform of the fourfold convolution

Directly,

$$\begin{aligned}\mathcal{F}[(f * f) * (f * f)](\omega) &= \frac{1}{2\pi}\int_0^{\infty} \frac{x^3}{6}e^{-(1+i\omega)x}\,dx \\ &= \frac{1}{2\pi}\frac{1}{(1+i\omega)^4}.\end{aligned}$$

Therefore

$$\boxed{\mathcal{F}[(f * f) * (f * f)](\omega) = \frac{1}{2\pi(1+i\omega)^4}}.$$

(d) Evaluation of an integral

Set

$$h(x) = \frac{x^3}{6}e^{-x}\mathbf{1}_{(0,\infty)}(x).$$

By part (c),

$$|\widehat{h}(\omega)|^2 = \frac{1}{4\pi^2(1+\omega^2)^4}.$$

Parseval's identity gives

$$\int_0^{\infty} \frac{x^6}{36}e^{-2x}\,dx = 2\pi\int_{-\infty}^{\infty} \frac{d\omega}{4\pi^2(1+\omega^2)^4}.$$

Since

$$\int_0^{\infty} x^6e^{-2x}\,dx = \frac{6!}{2^7},$$

we obtain

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^4} = \frac{5\pi}{16}}.$$

Exercise 4

Let $f = \mathbf{1}_{[-1,1]}$.

(a) Compute $f * f$

The value $(f * f)(x)$ is the length of the intersection $[-1, 1] \cap [x - 1, x + 1]$. Consequently,

$$\boxed{(f * f)(x) = \begin{cases} 2 - |x|, & |x| \le 2, \\ 0, & |x| > 2. \end{cases}}$$

(b) Two integrals

First,

$$\widehat{f}(\omega) = \frac{1}{2\pi} \int_{-1}^{1} e^{-i\omega x}\, dx = \frac{\sin \omega}{\pi \omega}.$$

Parseval's identity gives

$$2 = \int_{\mathbb{R}} |f(x)|^2\, dx = 2\pi \int_{\mathbb{R}} \frac{\sin^2 \omega}{\pi^2 \omega^2}\, d\omega,$$

so

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi}.$$

Next, by the convolution theorem,

$$\widehat{f * f}(\omega) = 2\pi \widehat{f}(\omega)^2 = \frac{2 \sin^2 \omega}{\pi \omega^2}.$$

Moreover,

$$\int_{\mathbb{R}} |(f * f)(x)|^2\, dx = 2 \int_{0}^{2} (2 - x)^2\, dx = \frac{16}{3}.$$

Applying Parseval once more,

$$\frac{16}{3} = 2\pi \int_{\mathbb{R}} \frac{4 \sin^4 \omega}{\pi^2 \omega^4}\, d\omega,$$

and hence

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4}\, dx = \frac{2\pi}{3}}.$$

Exercise 5

For $a > 0$, let

$$f_a(x) = e^{-a|x|}, \qquad g_a(x) = \frac{2a}{x^2 + a^2}.$$

(a) Fourier transform of f_a

Since f_a is even,

$$\begin{aligned} \widehat{f_a}(\omega) &= \frac{1}{\pi} \int_{0}^{\infty} e^{-ax} \cos(\omega x)\, dx \\ &= \boxed{\frac{a}{\pi(a^2 + \omega^2)}}. \end{aligned}$$

(b) Fourier transform of g_a

Part (a), together with Fourier inversion, gives the transform pair

$$e^{-a|x|} \longleftrightarrow \frac{a}{\pi(a^2+\omega^2)}.$$

Equivalently,

$$\boxed{\widehat{g_a}(\omega) = e^{-a|\omega|}}.$$

It is convenient to write

$$k_a(x) = \frac{1}{x^2+a^2} = \frac{1}{2a} g_a(x), \qquad \widehat{k_a}(\omega) = \frac{1}{2a} e^{-a|\omega|}.$$

(c)

Suppose first that we ignore the condition $\int \varphi = 1$ and solve

$$(\varphi * k_4)(x) = k_7(x).$$

Taking Fourier transforms gives

$$2\pi\widehat{\varphi}(\omega)\frac{1}{8}e^{-4|\omega|} = \frac{1}{14}e^{-7|\omega|},$$

so

$$\widehat{\varphi}(\omega) = \frac{2}{7\pi}e^{-3|\omega|}.$$

Using part (b),

$$\varphi(x) = \frac{2}{7\pi} g_3(x) = \frac{12}{7\pi(x^2+9)}.$$

This is the unique integrable solution of the convolution equation, but

$$\int_{-\infty}^{\infty} \varphi(t)\,dt = \frac{12}{7\pi}\cdot\frac{\pi}{3} = \frac{4}{7} \neq 1.$$

Therefore no function satisfying both requirements exists:

$$\boxed{\text{There is no such } \varphi \in G(\mathbb{R}).}$$

The incompatibility can also be seen immediately by integrating the convolution equation over x.

(d)

The equation is now

$$(\varphi * k_7)(x) = k_4(x).$$

Taking Fourier transforms would require

$$2\pi\widehat{\varphi}(\omega)\frac{1}{14}e^{-7|\omega|} = \frac{1}{8}e^{-4|\omega|},$$

and hence

$$\widehat{\varphi}(\omega) = \frac{7}{8\pi}e^{3|\omega|}.$$

This cannot be the Fourier transform of an integrable function: the Fourier transform of an L^1 function is bounded and tends to zero at infinity, whereas the expression above is unbounded. Thus

$$\boxed{\text{There is no such } \varphi \in G(\mathbb{R}).}$$

Also, integrating the proposed equation would force $\int \varphi = 7/4$, already contradicting the required normalization.

Exercise 6

Let $f \in C(\mathbb{R}^n)$ be absolutely integrable, and define its n-dimensional Fourier transform by

$$\mathcal{F}[f](\omega_1, \ldots, \omega_n) = \frac{1}{(2\pi)^n} \int \cdots \int f(x_1, \ldots, x_n) e^{-i(\omega_1 x_1 + \cdots + \omega_n x_n)}\, dx_1 \cdots dx_n.$$

Assume

$$f(x_1, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n),$$

where each $f_j \in G(\mathbb{R})$. Prove that

$$\mathcal{F}[f](\omega_1, \ldots, \omega_n) = \mathcal{F}[f_1](\omega_1) \cdots \mathcal{F}[f_n](\omega_n).$$

Solution

Absolute integrability permits the use of Fubini's theorem. Therefore

$$\begin{aligned}
\mathcal{F}[f](\omega_1, \ldots, \omega_n) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^{n} \left(f_j(x_j) e^{-i\omega_j x_j} \right) dx_1 \cdots dx_n \\
&= \prod_{j=1}^{n} \left(\frac{1}{2\pi} \int_{\mathbb{R}} f_j(x_j) e^{-i\omega_j x_j}\, dx_j \right) \\
&= \prod_{j=1}^{n} \mathcal{F}[f_j](\omega_j).
\end{aligned}$$

Hence

$$\boxed{\mathcal{F}[f](\omega_1, \ldots, \omega_n) = \mathcal{F}[f_1](\omega_1)\mathcal{F}[f_2](\omega_2) \cdots \mathcal{F}[f_n](\omega_n)}.$$

Note on the source sheet. The displayed multidimensional definition in the source appears to use limits $-\pi$ to π, although the surrounding text speaks about functions on $\mathbb{R}^n$. The same factorization proof is valid with those finite limits; one simply replaces every integral over $\mathbb{R}$ by an integral over $[-\pi, \pi]$.

h̃ead4.1 Laplace Transform h̃eadWorked Solutions

Chapter 4, Section 1
Laplace Transform – Exercises and Solutions

This document was generated by **GPT-5.6 Sol** (extended) which translated and solved the exercises in the supplied section in our **Fourier Series and Laplace Transforms** course textbook (Hebrew):
https://www.samyzaf.com/technion/fourier/fourier.pdf
AI chatbots may make errors. If you find any, please send report to mail@samyzaf.com.

Throughout, the Laplace transform is

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

whenever the improper integral converges.

Exercise 1

Determine whether the function

$$f(t) = e^{t^2}$$

has a Laplace transform. If it does, find it and state its domain of definition. If not, explain why.

Solution

Let $s \in \mathbb{C}$ and write $\sigma = \Re s$. Then

$$\left| e^{-st} e^{t^2} \right| = e^{t^2 - \sigma t}.$$

For $t \geq 2|\sigma| + 1$, we have

$$t^2 - \sigma t \geq t^2 - |\sigma| t \geq \frac{t^2}{2}.$$

Hence the integrand does not even tend to zero; in fact,

$$\left| e^{-st} e^{t^2} \right| \geq e^{t^2/2} \longrightarrow \infty.$$

Therefore

$$\int_0^\infty e^{-st} e^{t^2}\, dt$$

diverges for every $s \in \mathbb{C}$. Thus e^{t^2} has no Laplace transform in the usual sense.

$$\boxed{\mathcal{L}\{e^{t^2}\}(s) \text{ does not exist for any } s \in \mathbb{C}.}$$

Exercise 2

Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \leq t \leq 2\pi, \\ 0, & t > 2\pi. \end{cases}$$

Solution

Since f has compact support, its Laplace transform exists for every $s \in \mathbb{C}$. We compute

$$\mathcal{L}\{f\}(s) = \int_0^{2\pi} e^{-st} \sin t \, dt.$$

Using

$$\int e^{-st} \sin t \, dt = -\frac{e^{-st}(s \sin t + \cos t)}{s^2+1},$$

we obtain, initially for $s \neq \pm i$,

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \left[-\frac{e^{-st}(s \sin t + \cos t)}{s^2+1} \right]_0^{2\pi} \\ &= \frac{1 - e^{-2\pi s}}{s^2+1}.\end{aligned}$$

The apparent singularities at $s = \pm i$ are removable because the original integral is entire in s. Therefore

$$\boxed{\mathcal{L}\{f\}(s) = \frac{1 - e^{-2\pi s}}{s^2+1}, \qquad s \in \mathbb{C},}$$

with the values at $s = \pm i$ understood by continuity. Directly,

$$\mathcal{L}\{f\}(i) = -i\pi, \qquad \mathcal{L}\{f\}(-i) = i\pi.$$

Exercise 3

Suppose $f : [0, \infty) \to \mathbb{C}$ is piecewise continuous and periodic with period $p > 0$. Prove that

$$\mathcal{L}\{f\}(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) \, dt.$$

Solution

A piecewise continuous function on the compact interval $[0, p]$ is bounded there. Periodicity therefore implies that f is bounded on all of $[0, \infty)$. Hence the Laplace transform converges absolutely whenever $\Re s > 0$.

Let $\sigma = \Re s > 0$. Split the integral into periods:

$$\mathcal{L}\{f\}(s) = \sum_{k=0}^{\infty} \int_{kp}^{(k+1)p} e^{-st} f(t) \, dt.$$

In the kth integral, set $t = u + kp$. Since $f(u + kp) = f(u)$,

$$\int_{kp}^{(k+1)p} e^{-st} f(t) \, dt = e^{-kps} \int_0^p e^{-su} f(u) \, du.$$

Consequently,

$$\mathcal{L}\{f\}(s) = \left(\sum_{k=0}^{\infty} e^{-kps} \right) \int_0^p e^{-su} f(u) \, du.$$

Because $|e^{-ps}| = e^{-p\sigma} < 1$, the geometric series equals $(1 - e^{-ps})^{-1}$. Therefore

$$\boxed{\mathcal{L}\{f\}(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) \, dt, \qquad \Re s > 0.}$$

Exercise 4

Let $f : [0, \infty) \to \mathbb{R}$ be periodic with period 4 and, on one period, let

$$f(t) = \begin{cases} 1, & 0 \le t \le 2, \\ -1, & 2 < t \le 4. \end{cases}$$

Prove that

$$\mathcal{L}\{f\}(s) = \frac{\tanh s}{s}.$$

Solution

By Exercise 3, for $\Re s > 0$,

$$\mathcal{L}\{f\}(s) = \frac{1}{1 - e^{-4s}} \int_0^4 e^{-st} f(t)\, dt.$$

Now

$$\begin{aligned} \int_0^4 e^{-st} f(t)\, dt &= \int_0^2 e^{-st}\, dt - \int_2^4 e^{-st}\, dt \\ &= \frac{1 - e^{-2s}}{s} - \frac{e^{-2s} - e^{-4s}}{s} \\ &= \frac{(1 - e^{-2s})^2}{s}. \end{aligned}$$

Since

$$1 - e^{-4s} = (1 - e^{-2s})(1 + e^{-2s}),$$

we obtain

$$\begin{aligned} \mathcal{L}\{f\}(s) &= \frac{1 - e^{-2s}}{s(1 + e^{-2s})} \\ &= \frac{e^{s} - e^{-s}}{s(e^{s} + e^{-s})} \\ &= \frac{\tanh s}{s}. \end{aligned}$$

Thus

$$\boxed{\mathcal{L}\{f\}(s) = \frac{\tanh s}{s}, \qquad \Re s > 0.}$$

Editorial note on Exercise 4

In the supplied sheet, both branches of the piecewise definition are printed as 1. Taken literally, that makes $f(t) \equiv 1$, whose Laplace transform is $1/s$. The stated result $\tanh(s)/s$ is obtained when the second branch is -1, which is evidently the intended formulation.

Chapter 4, Section 2: Laplace Transforms

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Throughout, we use

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

whenever the integral converges. We repeatedly use

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a), \qquad \mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s).$$

Exercise 1

Compute the Laplace transform of each function.

(a) $e^{-t}\cos 2t$

Since

$$\mathcal{L}\{\cos 2t\}(s) = \frac{s}{s^2+4},$$

the exponential shift rule gives

$$\boxed{\mathcal{L}\{e^{-t}\cos 2t\}(s) = \frac{s+1}{(s+1)^2+4}}, \qquad \Re s > -1.$$

(b) $e^{-4t}\cosh 2t$

Using

$$\mathcal{L}\{\cosh 2t\}(s) = \frac{s}{s^2-4},$$

we obtain

$$\boxed{\mathcal{L}\{e^{-4t}\cosh 2t\}(s) = \frac{s+4}{(s+4)^2-4}}, \qquad \Re s > -2.$$

(c) $(t^2+1)^2$

Expand first:

$$(t^2+1)^2 = t^4 + 2t^2 + 1.$$

Since $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$,

$$\boxed{\mathcal{L}\{(t^2+1)^2\}(s) = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}}, \qquad \Re s > 0.$$

(d) $3\cosh t - 4\sinh 5t$

By linearity,

$$\mathcal{L}\{3\cosh t - 4\sinh 5t\}(s) = 3\frac{s}{s^2-1} - 4\frac{5}{s^2-25}.$$

Thus

$$\boxed{\mathcal{L}\{3\cosh t - 4\sinh 5t\}(s) = \frac{3s}{s^2-1} - \frac{20}{s^2-25}}, \qquad \Re s > 5.$$

(e) $t^n \sin t$

Let

$$F(s) = \mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}.$$

Then

$$\boxed{\mathcal{L}\{t^n \sin t\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s^2+1}\right)}.$$

An equivalent closed form is

$$\boxed{\mathcal{L}\{t^n \sin t\}(s) = \frac{n!}{2i}\left[\frac{1}{(s-i)^{n+1}} - \frac{1}{(s+i)^{n+1}}\right]}, \qquad \Re s > 0.$$

Exercise 2

Compute the Laplace transform of

$$f(t) = \frac{\sin t}{t}.$$

Define

$$I(a,s) = \int_0^\infty e^{-st}\frac{\sin(at)}{t}\,dt, \qquad s > 0.$$

Differentiate with respect to a:

$$\frac{\partial I}{\partial a} = \int_0^\infty e^{-st}\cos(at)\,dt = \frac{s}{s^2+a^2}.$$

Because $I(0,s) = 0$,

$$I(a,s) = \int_0^a \frac{s}{s^2+u^2}\,du = \arctan\left(\frac{a}{s}\right).$$

Setting $a = 1$ gives

$$\boxed{\mathcal{L}\left\{\frac{\sin t}{t}\right\}(s) = \arctan\left(\frac{1}{s}\right)}, \qquad s > 0.$$

Exercise 3

Given $a > 0$ and

$$\mathcal{L}\{f\}(s) = \frac{d^n}{ds^n}\left(\frac{1}{s^2-a^2}\right),$$

find f.

Since

$$\mathcal{L}\{\sinh(at)\}(s) = \frac{a}{s^2-a^2},$$

we have

$$\frac{1}{s^2-a^2} = \mathcal{L}\left\{\frac{1}{a}\sinh(at)\right\}(s).$$

Using

$$\frac{d^n}{ds^n}F(s) = (-1)^n\mathcal{L}\{t^n f(t)\}(s),$$

we conclude

$$\boxed{f(t) = \frac{(-1)^n}{a}\,t^n\sinh(at)}.$$

The transform is initially valid for $\Re s > a$.

Exercise 4

Prove each identity.

(a)

Using

$$\sin^2 t = \frac{1-\cos 2t}{2},$$

we get

$$\mathcal{L}\{\sin^2 t\}(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) = \boxed{\frac{2}{s(s^2+4)}}.$$

(b)

Expand the phase shift:

$$\cos(\omega t+\theta) = \cos\omega t\cos\theta - \sin\omega t\sin\theta.$$

Hence

$$\mathcal{L}\{A\cos(\omega t+\theta)\}(s) = A\left(\frac{s\cos\theta}{s^2+\omega^2} - \frac{\omega\sin\theta}{s^2+\omega^2}\right),$$

so

$$\boxed{\mathcal{L}\{A\cos(\omega t+\theta)\}(s) = \frac{A(s\cos\theta-\omega\sin\theta)}{s^2+\omega^2}}.$$

(c)

Since

$$\cosh(at) = \frac{e^{at}+e^{-at}}{2},$$

we obtain

$$\mathcal{L}\{\cos at\cosh at\}(s) = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right].$$

Putting the two terms over a common denominator gives

$$\boxed{\mathcal{L}\{\cos at\cosh at\}(s) = \frac{s^3}{s^4+4a^4}}, \qquad \Re s > |a|.$$

(d)

First,

$$\mathcal{L}\{t^2 - 5t + 6\}(s) = \frac{2}{s^3} - \frac{5}{s^2} + \frac{6}{s}.$$

The factor e^{2t} shifts s to $s-2$:

$$\mathcal{L}\{(t^2 - 5t + 6)e^{2t}\}(s) = \frac{2}{(s-2)^3} - \frac{5}{(s-2)^2} + \frac{6}{s-2}.$$

Combining terms,

$$\boxed{\mathcal{L}\{(t^2 - 5t + 6)e^{2t}\}(s) = \frac{6s^2 - 29s + 36}{(s-2)^3}}, \qquad \Re s > 2.$$

Exercise 5

Evaluate the integrals using Laplace transforms.

(a) $\int_0^\infty te^{-2t} \cos t \, dt$

Because

$$\mathcal{L}\{\cos t\}(s) = \frac{s}{s^2+1},$$

we have

$$\mathcal{L}\{t \cos t\}(s) = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}.$$

At $s = 2$,

$$\boxed{\int_0^\infty te^{-2t} \cos t \, dt = \frac{3}{25}}.$$

(b) $\int_0^\infty t^3 e^{-t} \sin t \, dt$

Using the complex form,

$$\int_0^\infty t^3 e^{-(1-i)t} \, dt = \frac{3!}{(1-i)^4}.$$

Now $(1-i)^2 = -2i$ and $(1-i)^4 = -4$, so the right-hand side is real. Therefore its imaginary part is zero:

$$\boxed{\int_0^\infty t^3 e^{-t} \sin t \, dt = 0}.$$

(c) $\int_0^\infty x^4 e^{-x} \, dx$

This is the Gamma integral:

$$\int_0^\infty x^4 e^{-x} \, dx = \frac{4!}{1^5}.$$

Hence

$$\boxed{\int_0^\infty x^4 e^{-x} \, dx = 24}.$$

(d) $\displaystyle\int_0^\infty x^6 e^{-3x}\,dx$

Similarly,

$$\int_0^\infty x^6 e^{-3x}\,dx = \frac{6!}{3^7} = \frac{720}{2187} = \boxed{\frac{80}{243}}.$$

Final Answers

$$\begin{aligned}
\mathcal{L}\{e^{-t}\cos 2t\}(s) &= \frac{s+1}{(s+1)^2+4},\\
\mathcal{L}\{e^{-4t}\cosh 2t\}(s) &= \frac{s+4}{(s+4)^2-4},\\
\mathcal{L}\{(t^2+1)^2\}(s) &= \frac{24}{s^5}+\frac{4}{s^3}+\frac{1}{s},\\
\mathcal{L}\{3\cosh t - 4\sinh 5t\}(s) &= \frac{3s}{s^2-1}-\frac{20}{s^2-25},\\
\mathcal{L}\{t^n \sin t\}(s) &= (-1)^n \frac{d^n}{ds^n}\frac{1}{s^2+1},\\
\mathcal{L}\left\{\frac{\sin t}{t}\right\}(s) &= \arctan\frac{1}{s},\\
f(t) &= \frac{(-1)^n}{a} t^n \sinh(at).
\end{aligned}$$

Chapter 4, Section 3
Inverse Laplace Transforms and Initial-Value Problems

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Throughout,

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Exercise 1

Find the inverse Laplace transform of each function.

(a) $F(s) = \dfrac{3s - 14}{s^2 - 4s + 18}$.

Solution. Complete the square:

$$s^2 - 4s + 18 = (s-2)^2 + 14, \qquad 3s - 14 = 3(s-2) - 8.$$

Using the shift rule,

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = e^{2t}\left(3\cos(\sqrt{14}\,t) - \frac{8}{\sqrt{14}}\sin(\sqrt{14}\,t)\right).}$$

(b) $F(s) = \dfrac{1}{s^2(s^2+1)}$.

Solution. Since

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1},$$

we obtain

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = t - \sin t.}$$

(c) $F(s) = \dfrac{1}{s^2 - 3s + 2}$.

Solution. Factor and decompose:

$$\frac{1}{(s-1)(s-2)} = -\frac{1}{s-1} + \frac{1}{s-2}.$$

Thus

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = e^{2t} - e^t.}$$

(d) $F(s) = \dfrac{s^2}{(s^2+4)^2}$.

Solution. Write

$$\frac{s^2}{(s^2+4)^2} = \frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}.$$

We use

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{\sin(at) - at\cos(at)}{2a^3}.$$

With $a = 2$,

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = \frac{1}{4}\sin 2t + \frac{1}{2}t\cos 2t.}$$

(e) $F(s) = \dfrac{s}{(s^2+4)^2}$.

Solution. Because

$$\mathcal{L}\{t\sin(at)\} = \frac{2as}{(s^2+a^2)^2},$$

setting $a = 2$ gives

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = \frac{1}{4}t\sin 2t.}$$

(f) $F(s) = \log\left(1 + \dfrac{1}{s^2}\right)$.

Solution. Define

$$I(a,s) = \int_0^\infty e^{-st}\frac{1-\cos(at)}{t}\,dt.$$

Differentiation with respect to a gives

$$\frac{\partial I}{\partial a} = \int_0^\infty e^{-st}\sin(at)\,dt = \frac{a}{s^2+a^2}.$$

Since $I(0,s) = 0$,

$$I(a,s) = \frac{1}{2}\log\left(1 + \frac{a^2}{s^2}\right).$$

Taking $a = 1$ yields

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = \frac{2(1-\cos t)}{t},}$$

with the removable value 0 at $t = 0$.

(g) $F(s) = \dfrac{1}{(s^2+4)^2}$.

Solution. Using the formula from part (d) with $a = 2$,

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = \frac{1}{16}\big(\sin 2t - 2t\cos 2t\big).}$$

(h) $F(s) = \dfrac{s^3}{s^4-16}$.

Solution. Since

$$\frac{s^3}{(s^2-4)(s^2+4)} = \frac{1}{2}\frac{s}{s^2-4} + \frac{1}{2}\frac{s}{s^2+4},$$

we find

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = \frac{1}{2}\cosh 2t + \frac{1}{2}\cos 2t.}$$

(i) $F(s) = \dfrac{1}{s^4+1}$.

Solution. Put $a = 1/\sqrt{2}$. Then

$$s^4 + 1 = (s^2 + \sqrt{2}s + 1)(s^2 - \sqrt{2}s + 1)$$

and

$$\frac{1}{s^4+1} = -\frac{\sqrt{2}s - 2}{4(s^2 - \sqrt{2}s + 1)} + \frac{\sqrt{2}s + 2}{4(s^2 + \sqrt{2}s + 1)}.$$

Completing the squares and applying the shift rule gives

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = \frac{1}{\sqrt{2}}\left[\cosh\left(\frac{t}{\sqrt{2}}\right)\sin\left(\frac{t}{\sqrt{2}}\right) - \sinh\left(\frac{t}{\sqrt{2}}\right)\cos\left(\frac{t}{\sqrt{2}}\right)\right].}$$

(j) $F(s) = \dfrac{2s^2 + s - 10}{(s-4)(s^2+2s+2)}$.

Solution. Partial fractions give

$$F(s) = \frac{1}{s-4} + \frac{s+3}{s^2+2s+2} = \frac{1}{s-4} + \frac{s+1}{(s+1)^2+1} + \frac{2}{(s+1)^2+1}.$$

Therefore

$$\boxed{\mathcal{L}^{-1}\{F\}(t) = e^{4t} + e^{-t}\cos t + 2e^{-t}\sin t.}$$

Exercise 2

For each differential equation, find the solution on $[0, \infty)$ satisfying the stated initial conditions. Although these equations can be solved directly, we display the Laplace-transform calculation.

(a) $y'' + 3y' - 4y = 0$, $y(0) = 3$, $y'(0) = -2$.

Solution. Let $Y = \mathcal{L}\{y\}$. Transforming gives

$$(s^2Y - 3s + 2) + 3(sY - 3) - 4Y = 0,$$

so

$$Y = \frac{3s+7}{(s-1)(s+4)} = \frac{2}{s-1} + \frac{1}{s+4}.$$

Hence

$$\boxed{y(t) = 2e^t + e^{-4t}.}$$

(b) $y'' - y = 0$, $y(0) = 1$, $y'(0) = -1$.

Solution. The transformed equation is

$$(s^2-1)Y - s + 1 = 0, \qquad Y = \frac{s-1}{(s-1)(s+1)} = \frac{1}{s+1}.$$

Thus

$$\boxed{y(t) = e^{-t}.}$$

(c) $y'' + 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution. We obtain

$$(s^2 + 4s + 4)Y - s - 4 = 0, \qquad Y = \frac{s+4}{(s+2)^2} = \frac{1}{s+2} + \frac{2}{(s+2)^2}.$$

Therefore

$$\boxed{y(t) = (1 + 2t)e^{-2t}.}$$

(d) $y'' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$.

Solution. Transforming gives

$$(s^2 + 4)Y - 2s - 1 = 0, \qquad Y = \frac{2s+1}{s^2+4}.$$

Consequently

$$\boxed{y(t) = 2\cos 2t + \frac{1}{2}\sin 2t.}$$

(e) $y'' + 4y' + 7y = 0$, $y(0) = 1$, $y'(0) = 1$.

Solution. The transformed equation is

$$(s^2 + 4s + 7)Y - s - 5 = 0.$$

Since $s^2 + 4s + 7 = (s+2)^2 + 3$ and $s + 5 = (s+2) + 3$,

$$Y = \frac{s+2}{(s+2)^2 + 3} + \frac{3}{(s+2)^2 + 3}.$$

Hence

$$\boxed{y(t) = e^{-2t}\big(\cos(\sqrt{3}\,t) + \sqrt{3}\sin(\sqrt{3}\,t)\big).}$$

(f) $ty''(t) + 2y'(t) + ty(t) = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution. Let $Y(s) = \mathcal{L}\{y\}(s)$. We use

$$\mathcal{L}\{tq(t)\} = -\frac{d}{ds}\mathcal{L}\{q(t)\}.$$

Since

$$\mathcal{L}\{y''\} = s^2Y - s, \qquad \mathcal{L}\{y'\} = sY - 1, \qquad \mathcal{L}\{ty\} = -Y',$$

we have

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}(s^2Y - s) = -2sY - s^2Y' + 1.$$

Therefore the transformed differential equation becomes

$$(-2sY - s^2Y' + 1) + 2(sY - 1) - Y' = 0,$$

or

$$(s^2 + 1)Y' = -1.$$

Thus

$$Y(s) = C - \arctan s.$$

Because every Laplace transform tends to 0 as $s \to \infty$ under the present hypotheses, $C = \pi/2$. Hence

$$Y(s) = \frac{\pi}{2} - \arctan s = \arctan\frac{1}{s}.$$

The standard identity

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\}(s) = \arctan\frac{1}{s}$$

gives

$$\boxed{y(t) = \frac{\sin t}{t}, \qquad y(0) := 1.}$$

The value at $t = 0$ is removable, and the resulting function also satisfies $y'(0) = 0$.

Chapter 4, Section 4
Laplace Transforms with Steps, Impulses, and Delays

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We use
$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt,$$
write $H(t-a)$ for the Heaviside step at a, and write $\delta_a(t) = \delta(t-a)$.

Exercise 1: Direct Laplace transforms

Compute the Laplace transform of each function.

(a)

$$f(t) = \begin{cases} 1, & 0 \le t \le T, \\ 0, & t > T. \end{cases}$$

Solution. Direct integration gives
$$\mathcal{L}\{f\}(s) = \int_0^T e^{-st}\,dt = \boxed{\frac{1-e^{-Ts}}{s}}.$$

(b)

$$f(t) = \begin{cases} 0, & 0 \le t \le 1, \\ (t-1)^2, & t > 1. \end{cases}$$

Solution. Since $f(t) = H(t-1)(t-1)^2$, the second shifting theorem yields
$$\boxed{\mathcal{L}\{f\}(s) = e^{-s}\frac{2}{s^3}}.$$

(c)

$$f(t) = \begin{cases} t^2, & 0 \le t < 2, \\ 4t, & t \ge 2. \end{cases}$$

Solution. Write
$$f(t) = t^2 + H(t-2)\big(4t - t^2\big).$$

For $u = t - 2$,

$$4t - t^2 = 4(u+2) - (u+2)^2 = 4 - u^2.$$

Therefore

$$\boxed{\mathcal{L}\{f\}(s) = \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)}.$$

(d)

$$f(t) = \begin{cases} t, & 0 \le t \le 7, \\ 3 - t, & 7 < t < 8, \\ 1, & t \ge 8. \end{cases}$$

Solution. Starting from t, the correction at $t = 7$ is

$$(3 - t) - t = 3 - 2t = -11 - 2(t - 7),$$

and the correction at $t = 8$ is

$$1 - (3 - t) = t - 2 = 6 + (t - 8).$$

Thus

$$f(t) = t + H(t-7)\big[-11 - 2(t-7)\big] + H(t-8)\big[6 + (t-8)\big].$$

Hence

$$\boxed{\mathcal{L}\{f\}(s) = \frac{1}{s^2} + e^{-7s}\left(-\frac{11}{s} - \frac{2}{s^2}\right) + e^{-8s}\left(\frac{6}{s} + \frac{1}{s^2}\right).}$$

Exercise 2: Inverse Laplace transforms

(a)

$$F(s) = \frac{e^{-s}(1 - e^{-s})}{s(s^2+1)}.$$

Solution. First,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1},$$

so its inverse transform is $1 - \cos t$. Since

$$F(s) = e^{-s}A(s) - e^{-2s}A(s),$$

we obtain

$$\boxed{f(t) = H(t-1)\big[1 - \cos(t-1)\big] - H(t-2)\big[1 - \cos(t-2)\big].}$$

(b)

$$F(s) = \frac{e^{-16s}}{s(s^2+2s+4)}.$$

Solution. Since $s^2+2s+4=(s+1)^2+3$,

$$\frac{1}{s(s^2+2s+4)} = \frac{1}{4s} - \frac{s+1}{4((s+1)^2+3)} - \frac{1}{4((s+1)^2+3)}.$$

Therefore

$$r(t) = \frac{1}{4} - \frac{1}{4}e^{-t}\cos(\sqrt{3}t) - \frac{1}{4\sqrt{3}}e^{-t}\sin(\sqrt{3}t).$$

Applying the delay by 16,

$$\boxed{f(t) = H(t-16)\,r(t-16).}$$

Exercise 3: A delta distribution

Prove that

$$\mathcal{L}\{\cos t\,\ln t\,\delta_\pi(t)\}(s) = -e^{-\pi s}\ln\pi.$$

Solution. The sampling property of the Dirac delta gives

$$\int_0^\infty e^{-st}\cos t\,\ln t\,\delta(t-\pi)\,dt = e^{-\pi s}\cos\pi\,\ln\pi.$$

Since $\cos\pi = -1$,

$$\boxed{\mathcal{L}\{\cos t\ln t\,\delta_\pi(t)\}(s) = -e^{-\pi s}\ln\pi.}$$

Exercise 4: Initial-value problems

(a)

$$y'' + 4y' + 7y = H(t-1), \qquad y(0)=1, \quad y'(0)=0.$$

Solution. Taking transforms,

$$(s^2+4s+7)Y - (s+4) = \frac{e^{-s}}{s}.$$

The zero-input part is

$$y_0(t) = e^{-2t}\left(\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right).$$

Also

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+7)}\right\} = \frac{1}{7} - \frac{1}{7}e^{-2t}\cos(\sqrt{3}t) - \frac{2}{7\sqrt{3}}e^{-2t}\sin(\sqrt{3}t).$$

Thus

$$\boxed{y(t) = y_0(t) + H(t-1)r(t-1),}$$

where

$$r(t) = \frac{1}{7}\left[1 - e^{-2t}\cos(\sqrt{3}t) - \frac{2}{\sqrt{3}}e^{-2t}\sin(\sqrt{3}t)\right].$$

(b)

$$y'' + 2y' + 3y = \delta_\pi(t), \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution. The homogeneous solution satisfying the initial data is

$$y_0(t) = e^{-t}\left(\cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)\right).$$

The impulse response is

$$h(t) = \frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t).$$

Hence

$$\boxed{y(t) = y_0(t) + \frac{1}{\sqrt{2}}H(t-\pi)e^{-(t-\pi)}\sin\big(\sqrt{2}(t-\pi)\big).}$$

(c)

$$y'' - 2y' + y = (-1)^{\lfloor t \rfloor}, \qquad y(0) = y'(0) = 0.$$

Solution. Let $q(t) = (-1)^{\lfloor t \rfloor}$. It is periodic with period 2, and

$$\mathcal{L}\{q\}(s) = \frac{1-e^{-s}}{s(1+e^{-s})} = \frac{1}{s}\tanh\frac{s}{2}.$$

Since the transfer function of $(D-1)^2$ is $1/(s-1)^2$,

$$Y(s) = \frac{1-e^{-s}}{s(1+e^{-s})(s-1)^2}.$$

Equivalently, using the impulse response $(t)e^t$,

$$\boxed{y(t) = \int_0^t (t-\tau)e^{t-\tau}(-1)^{\lfloor \tau \rfloor}\,d\tau.}$$

This formula is a finite elementary sum on every interval $n \le t < n+1$ and satisfies both initial conditions.

(d)

$$y'' + 2y' + 2y = \delta_\pi(t), \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution. The initial-data solution is $e^{-t}(\cos t + \sin t)$, while the impulse response is $e^{-t}\sin t$. Therefore

$$\boxed{y(t) = e^{-t}(\cos t + \sin t) + H(t-\pi)e^{-(t-\pi)}\sin(t-\pi).}$$

(e)

$$y'' + 4y = \delta_\pi(t) - \delta_{2\pi}(t), \qquad y(0) = y'(0) = 0.$$

Solution. The impulse response is $\frac{1}{2}\sin(2t)$. Hence

$$\boxed{y(t) = \frac{1}{2}H(t-\pi)\sin\big(2(t-\pi)\big) - \frac{1}{2}H(t-2\pi)\sin\big(2(t-2\pi)\big).}$$

(f)

$$y''' - y = h(t), \qquad y(0) = y'(0) = 0, \quad y''(0) = 1,$$

where $h(t) = 1$ for $\pi \le t \le 2\pi$ and $h(t) = 0$ otherwise.

Solution. We have

$$\mathcal{L}\{h\}(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s},$$

and the transformed equation is

$$(s^3 - 1)Y(s) - 1 = \frac{e^{-\pi s} - e^{-2\pi s}}{s}.$$

Define

$$r(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3 - 1}\right\} = \frac{1}{3}e^t - \frac{1}{3}e^{-t/2}\cos\frac{\sqrt{3}t}{2} - \frac{1}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}t}{2},$$

and

$$q(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^3 - 1)}\right\} = \int_0^t r(u)\,du.$$

Then

$$\boxed{y(t) = r(t) + H(t-\pi)q(t-\pi) - H(t-2\pi)q(t-2\pi).}$$

This form displays separately the initial-data response and the responses to switching the forcing on at π and off at 2π.

Exercise 5: A delay differential equation

Solve

$$y'(t) + y(t-1) = t^2, \qquad t > 0,$$

with history $y(t) = 0$ for $t \le 0$.

Solution. Because the prescribed history is zero,

$$\mathcal{L}\{y(t-1)\}(s) = e^{-s}Y(s), \qquad \mathcal{L}\{y'\}(s) = sY(s).$$

Thus

$$(s + e^{-s})Y(s) = \frac{2}{s^3}, \qquad Y(s) = \frac{2}{s^4}\frac{1}{1 + e^{-s}/s}.$$

Expanding as a geometric series,

$$Y(s) = 2\sum_{n=0}^{\infty} \frac{(-1)^n e^{-ns}}{s^{n+4}}.$$

After inversion,

$$\boxed{y(t) = 2\sum_{n=0}^{\infty}(-1)^n H(t-n)\frac{(t-n)^{n+3}}{(n+3)!}.}$$

For each fixed t, only finitely many terms are nonzero. In particular,

$$y(t) = \frac{t^3}{3}, \quad 0 \le t < 1,$$

and the later intervals are generated automatically by the finite sum (the method of steps).

Chapter 4, Section 5
Integral Equations and Initial-Value Problems

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Throughout, $F(s) = \mathcal{L}\{f\}(s)$ and convolution is on $[0, \infty)$:

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du.$$

We repeatedly use $\mathcal{L}\{f * g\} = FG$, $\mathcal{L}\{f'\} = sF - f(0)$, and the usual shift rule.

Exercise 1

Compute the Laplace transform of

$$f(t) = \int_0^t \left(u^2 - u + e^{-u}\right) du.$$

Solution. Since integration from 0 to t divides the transform by s,

$$\mathcal{L}\{f\}(s) = \frac{1}{s}\left(\frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{s+1}\right) = \boxed{\frac{2}{s^4} - \frac{1}{s^3} + \frac{1}{s(s+1)}}.$$

Exercise 2

Solve the following integral equations.

(a)

$$f(t) + 2\int_0^t f(u)\cos(t-u)\,du = 9e^{2t}.$$

Solution.

$$F\left(1 + \frac{2s}{s^2+1}\right) = \frac{9}{s-2}, \qquad F = \frac{9(s^2+1)}{(s-2)(s+1)^2}.$$

Partial fractions give

$$F = \frac{5}{s-2} + \frac{4}{s+1} - \frac{6}{(s+1)^2},$$

therefore

$$\boxed{f(t) = 5e^{2t} + 4e^{-t} - 6te^{-t}}.$$

(b)

For $a > 0$,

$$\int_0^t f(u)\,du - f'(t) = H(t-a).$$

Solution. At $t = 0$ the equation gives $f'(0) = 0$, but it does not determine $f(0)$. If $C = f(0)$, then

$$\frac{F}{s} - (sF - C) = \frac{e^{-as}}{s},$$

so

$$F(s) = \frac{e^{-as} - Cs}{1 - s^2}.$$

Equivalently, before the jump one has $f(t) = C\cosh t$, and the unit jump in the right side causes a jump -1 in f'. Thus

$$\boxed{f(t) = C\cosh t - H(t-a)\sinh(t-a), \qquad C \in \mathbb{R}.}$$

Hence the problem, as printed, has a one-parameter family of solutions.

(c)

$$f(t) + \int_0^t (t-u)f(u)\,du = \sin 2t.$$

Solution. Since $\mathcal{L}\{t\} = 1/s^2$,

$$F\left(1 + \frac{1}{s^2}\right) = \frac{2}{s^2+4}, \qquad F = \frac{2s^2}{(s^2+1)(s^2+4)}.$$

Thus

$$F = -\frac{2}{3(s^2+1)} + \frac{8}{3(s^2+4)},$$

and

$$\boxed{f(t) = -\frac{2}{3}\sin t + \frac{4}{3}\sin 2t}.$$

(d)

$$f''(t) = \int_0^t u f(t-u)\,du, \qquad f(0) = -1, \qquad f'(0) = 1.$$

Solution. Transforming,

$$s^2F + s - 1 = \frac{F}{s^2}, \qquad F = \frac{s^2(1-s)}{s^4-1}.$$

After cancellation and partial fractions,

$$F = -\frac{s-1}{2(s^2+1)} - \frac{1}{2(s+1)}.$$

Hence

$$\boxed{f(t) = \frac{1}{2}\sin t - \frac{1}{2}\cos t - \frac{1}{2}e^{-t}}.$$

(e)

$$f(t) + \int_0^t f(u)e^{-(t-u)}\,du = 1.$$

Solution.

$$F\left(1 + \frac{1}{s+1}\right) = \frac{1}{s}, \qquad F = \frac{s+1}{s(s+2)} = \frac{1}{2s} + \frac{1}{2(s+2)}.$$

Therefore

$$\boxed{f(t) = \frac{1}{2}\big(1 + e^{-2t}\big)}.$$

(f)

$$\int_0^t f'(u)f(t-u)\,du = 3te^{3t} - e^{3t} + 1, \qquad f(0) = 0, \qquad f'(0) > 0.$$

Solution. The transform of the left side is sF^2. The right side has transform

$$\frac{3}{(s-3)^2} - \frac{1}{s-3} + \frac{1}{s} = \frac{9}{s(s-3)^2}.$$

Thus

$$F^2 = \frac{9}{s^2(s-3)^2}, \qquad F = \pm\frac{3}{s(s-3)}.$$

The condition $f'(0) > 0$ selects the plus sign, so

$$F = \frac{1}{s-3} - \frac{1}{s}, \qquad \boxed{f(t) = e^{3t} - 1}.$$

(g)

$$3f'(t) - 10f(t) + 3\int_0^t f(u)\,du = 10\sin t - 5, \qquad f(0) = 2.$$

Solution.

$$(3s - 10 + 3/s)F - 6 = \frac{10}{s^2+1} - \frac{5}{s}.$$

Therefore

$$F = \frac{2s^2 - s + 5}{(s-3)(s^2+1)} = \frac{2}{s-3} - \frac{1}{s^2+1},$$

and

$$\boxed{f(t) = 2e^{3t} - \sin t}.$$

(h)

$$\int_0^t f(u)f(t-u)\,du = 2f(t) + t - 2.$$

Is the solution unique?

Solution.

$$F^2 = 2F + \frac{1}{s^2} - \frac{2}{s},$$

so

$$(F-1)^2 = \left(1 - \frac{1}{s}\right)^2.$$

Hence either $F = 1/s$, or $F = 2 - 1/s$. The second expression contains a nonzero constant term and is not the Laplace transform of an ordinary locally integrable function; it corresponds to a distribution containing a Dirac delta. In the usual function class,

$$\boxed{F = \frac{1}{s}, \qquad f(t) = 1,}$$

and the solution is unique.

(i)

$$f'(t) + \int_0^t f(u)\,du = \sin t, \qquad f(0) = 1.$$

Solution.

$$(s + 1/s)F - 1 = \frac{1}{s^2+1}, \qquad F = \frac{s(s^2+2)}{(s^2+1)^2}.$$

Using

$$\mathcal{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2},$$

we obtain

$$\boxed{f(t) = \cos t + \frac{1}{2}t\sin t}.$$

(j)

Assuming the intended convolution form

$$\int_0^t f''(u)f(t-u)\,du = te^{at}, \qquad f(0) = \frac{1}{a}, \qquad f'(0) = 1,$$

find f.

Solution. The function

$$\boxed{f(t) = \frac{1}{a}e^{at}}$$

satisfies the initial conditions, and

$$f''(u)f(t-u) = ae^{au} \cdot \frac{1}{a}e^{a(t-u)} = e^{at}.$$

Therefore the integral equals te^{at}, as required. The printed integrand appears to contain $f''(t)$ rather than $f''(u)$; the latter is the natural convolution form used here.

(k)

$$f(t) = at + \int_0^t f(u)\sin(t-u)\,du.$$

Solution.

$$F = \frac{a}{s^2} + \frac{F}{s^2+1},$$

so

$$F = \frac{a(s^2+1)}{s^4} = \frac{a}{s^2} + \frac{a}{s^4}.$$

Hence

$$\boxed{f(t) = a\left(t + \frac{t^3}{6}\right)}.$$

(l)

$$f'(t) + 5\int_0^t f(u)\cos 2(t-u)\,du = 10, \qquad f(0) = 2.$$

Solution.

$$(sF-2) + 5\frac{s}{s^2+4}F = \frac{10}{s}.$$

Thus

$$F = \frac{2(s+5)(s^2+4)}{s^2(s^2+9)} = \frac{8}{9s} + \frac{40}{9s^2} + \frac{10(s+5)}{9(s^2+9)}.$$

Therefore

$$\boxed{f(t) = \frac{8}{9} + \frac{40}{9}t + \frac{10}{9}\cos 3t + \frac{50}{27}\sin 3t}.$$

(m)

$$\int_0^t f'(u)f(t-u)\,du = 24t^3, \qquad f(0) = 0.$$

Solution.

$$sF^2 = \frac{144}{s^4}, \qquad F = \pm\frac{12}{s^{5/2}}.$$

Since $\mathcal{L}\{t^{3/2}\} = \Gamma(5/2)s^{-5/2}$ and $\Gamma(5/2) = 3\sqrt{\pi}/4$,

$$\boxed{f(t) = \pm\frac{16}{\sqrt{\pi}}t^{3/2}}.$$

Both signs satisfy the equation, so uniqueness fails.

Exercise 3

Solve each initial-value problem on $[0,\infty)$.

(a)

$$y'' + y = g(t), \qquad y(0) = y'(0) = 0,$$

where $g(t) = t$ for $0 \le t < 1$ and $g(t) = 1$ for $t \ge 1$.

Solution. Write

$$g(t) = t - (t-1)H(t-1).$$

Since $\mathcal{L}^{-1}\{1/[s^2(s^2+1)]\} = t - \sin t$,

$$\boxed{y(t) = t - \sin t - H(t-1)\big[(t-1) - \sin(t-1)\big]}.$$

(b)

$$y'' + 2y' - 3y = f(t), \qquad y(0) = 1, \quad y'(0) = 0,$$

where $f(t) = 0$ for $0 \le t \le 2\pi$ and $f(t) = \sin t$ for $t > 2\pi$.

Solution. Because $\sin(t) = \sin(t - 2\pi)$,

$$f(t) = H(t-2\pi)\sin(t-2\pi).$$

The zero-input part is

$$y_0(t) = \frac{3}{4}e^t + \frac{1}{4}e^{-3t}.$$

Furthermore,

$$q(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s-1)(s+3)}\right\} = \frac{1}{8}e^t - \frac{1}{40}e^{-3t} - \frac{1}{5}\sin t - \frac{1}{10}\cos t.$$

Hence

$$\boxed{y(t) = \frac{3}{4}e^t + \frac{1}{4}e^{-3t} + H(t-2\pi)q(t-2\pi)}.$$

(c)

$$y'' + 2y' + 3y = \delta_\pi(t) + \sin t, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Solution. Let

$$k(t) = \frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t).$$

This is the impulse response of $D^2 + 2D + 3$. The initial velocity contributes another copy of k. The sinusoidal forcing contributes

$$r(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+2s+3)}\right\} = -\frac{\sqrt{2}}{4}\cos\left(t + \frac{\pi}{4}\right) + \frac{1}{4}e^{-t}\cos(\sqrt{2}t).$$

Therefore

$$\boxed{y(t) = k(t) + r(t) + H(t-\pi)k(t-\pi)}.$$

(d)

$$y'' + y = \delta_\pi(t)\cos t, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Solution. Since $\delta_\pi(t)\cos t = \cos\pi\,\delta_\pi(t) = -\delta_\pi(t)$,

$$Y = \frac{1 - e^{-\pi s}}{s^2 + 1}.$$

Thus

$$\boxed{y(t) = \sin t - H(t-\pi)\sin(t-\pi)}.$$

(e)

$$y''' - y'' + 4y' - 4y = 68e^t \sin 2t,$$
$$y(0) = 1, \qquad y'(0) = -19, \qquad y''(0) = -37.$$

Solution. Transforming and decomposing gives

$$Y = -\frac{2(s+7)}{(s-1)^2 + 4} + \frac{2(7s-3)}{5(s^2+4)} + \frac{1}{5(s-1)}.$$

Therefore

$$\boxed{y(t) = -8e^t \sin 2t - 2e^t \cos 2t + \frac{1}{5}e^t - \frac{3}{5}\sin 2t + \frac{14}{5}\cos 2t.}$$

Direct substitution verifies both the equation and the three initial conditions.

(f)

$$y'' + 9y = f_c(t), \qquad y(0) = a, \quad y'(0) = b,$$

where $c > 0$ and

$$f_c(t) = \begin{cases} 0, & 0 \le t \le c, \\ t - c, & t > c. \end{cases}$$

Solution. Since $f_c(t) = H(t-c)(t-c)$,

$$\boxed{y(t) = a\cos 3t + \frac{b}{3}\sin 3t + H(t-c)\left(\frac{t-c}{9} - \frac{\sin 3(t-c)}{27}\right).}$$

Exercise 4

Let y_1, y_2 solve

$$y'' - 4y' + 4y = f_j(t), \qquad y(0) = 1, \qquad y'(0) = 1,$$

with $f_1(t) \le f_2(t)$ for all $t \ge 0$. Prove $y_1(t) \le y_2(t)$.

Solution. Set $z = y_2 - y_1$. Then

$$z'' - 4z' + 4z = f_2 - f_1, \qquad z(0) = z'(0) = 0.$$

The impulse response of $(D-2)^2$ is

$$k(t) = te^{2t}H(t),$$

which is nonnegative for $t \geq 0$. Hence

$$z(t) = \int_0^t (t-u)e^{2(t-u)}\,[f_2(u) - f_1(u)]\,du \geq 0.$$

Therefore

$$\boxed{y_1(t) \leq y_2(t) \qquad (t \geq 0).}$$

Editorial notes

- Exercise 2(b) lacks an initial value for $f(0)$, so the solution is not unique.
- Exercise 2(j) appears to print $f''(t)$ inside the integral; the natural convolution form is $f''(u)$.
- Exercise 2(h) is unique in the ordinary-function class, but admits a second distributional branch.

Chapter 4, Section 6
Beta and Gamma Functions

Throughout,

$$\mathrm{B}(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \qquad \Gamma(p)=\int_0^\infty t^{p-1}e^{-t}\,dt.$$

Exercise 1

Solve the integral equations.

(a)

$$\int_0^t \frac{f(u)}{\sqrt{t-u}}\,du=1+t+t^2.$$

This is a convolution with $k(t)=t^{-1/2}$. Since

$$\mathcal{L}\{k\}(s)=\Gamma\left(\frac{1}{2}\right)s^{-1/2}=\sqrt{\pi}\,s^{-1/2},$$

we obtain

$$F(s)\sqrt{\pi}\,s^{-1/2}=\frac{1}{s}+\frac{1}{s^2}+\frac{2}{s^3}.$$

Hence

$$F(s)=\frac{1}{\sqrt{\pi}}\left(s^{-1/2}+s^{-3/2}+2s^{-5/2}\right).$$

Using

$$\mathcal{L}^{-1}\{s^{-\alpha}\}(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

we get

$$\boxed{f(t)=\frac{1}{\pi\sqrt{t}}+\frac{2\sqrt{t}}{\pi}+\frac{8t^{3/2}}{3\pi}, \qquad t>0.}$$

(b)

$$\int_0^t \frac{f(u)}{\sqrt[3]{t-u}}\,du=t(1+t).$$

Here $k(t)=t^{-1/3}=t^{2/3-1}$, so

$$\mathcal{L}\{k\}(s)=\Gamma\left(\frac{2}{3}\right)s^{-2/3}.$$

Therefore

$$F(s)\Gamma\left(\frac{2}{3}\right)s^{-2/3}=\frac{1}{s^2}+\frac{2}{s^3},$$

and

$$F(s) = \frac{1}{\Gamma(2/3)}\left(s^{-4/3} + 2s^{-7/3}\right).$$

Consequently,

$$\boxed{f(t) = \frac{t^{1/3}}{\Gamma(2/3)\Gamma(4/3)} + \frac{2t^{4/3}}{\Gamma(2/3)\Gamma(7/3)}, \qquad t > 0.}$$

Exercise 2

Prove that for $p, q > 0$,

$$2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta = \mathrm{B}(p, q).$$

Set $x = \sin^2\theta$. Then

$$dx = 2\sin\theta\cos\theta \, d\theta,$$

and

$$\sin^{2p-1}\theta\cos^{2q-1}\theta \, d\theta = \frac{1}{2}x^{p-1}(1-x)^{q-1}\,dx.$$

As θ goes from 0 to $\pi/2$, x goes from 0 to 1. Thus

$$2\int_0^{\pi/2} \sin^{2p-1}\theta\cos^{2q-1}\theta \, d\theta = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \boxed{\mathrm{B}(p, q)}.$$

Exercise 3

Compute the following integrals.

(a)

$$\int_0^\infty x^2 e^{-2x^2}\,dx.$$

With $u = 2x^2$,

$$\int_0^\infty x^2 e^{-2x^2}\,dx = \frac{1}{2^{5/2}}\Gamma\left(\frac{3}{2}\right) = \boxed{\frac{\sqrt{\pi}}{8\sqrt{2}}}.$$

(b)

$$\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\,dx.$$

Set $u = \sqrt{x}$, so $x = u^2$, $dx = 2u\,du$, and $x^{1/4} = u^{1/2}$. Hence

$$\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\,dx = 2\int_0^\infty u^{3/2}e^{-u}\,du = 2\Gamma\left(\frac{5}{2}\right) = \boxed{\frac{3\sqrt{\pi}}{2}}.$$

Exercise 4

Prove that for $p > 0$,

$$\int_0^1 \left(\ln\frac{1}{x}\right)^{p-1}\,dx = \Gamma(p).$$

Let $t = \ln(1/x) = -\ln x$. Then $x = e^{-t}$ and $dx = -e^{-t}dt$. Therefore

$$\int_0^1 \left(\ln\frac{1}{x}\right)^{p-1}\,dx = \int_0^\infty t^{p-1}e^{-t}\,dt = \boxed{\Gamma(p)}.$$

Exercise 5

For $p, q > -1$, prove that
$$\int_0^1 x^p \left(\ln \frac{1}{x}\right)^q dx = \frac{\Gamma(q+1)}{(p+1)^{q+1}}.$$
Again set $t = \ln(1/x)$. Then $x^p = e^{-pt}$ and $dx = e^{-t}dt$ after reversing the limits. Thus
$$\int_0^1 x^p \left(\ln \frac{1}{x}\right)^q dx = \int_0^\infty t^q e^{-(p+1)t}\, dt.$$
Now let $u = (p+1)t$. We obtain
$$\boxed{\int_0^1 x^p \left(\ln \frac{1}{x}\right)^q dx = \frac{1}{(p+1)^{q+1}} \int_0^\infty u^q e^{-u}\, du = \frac{\Gamma(q+1)}{(p+1)^{q+1}}.}$$

Exercise 6

Compute the following integrals.

(a)

$$\int_0^2 (4-x^2)^{3/2}\, dx.$$
Set $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$ and
$$(4-x^2)^{3/2} = 8\cos^3\theta.$$
Therefore
$$\int_0^2 (4-x^2)^{3/2}\, dx = 16 \int_0^{\pi/2} \cos^4\theta\, d\theta = 16 \cdot \frac{3\pi}{16} = \boxed{3\pi}.$$

(b)

$$\int_0^\infty \frac{1-\cos x}{x^2}\, dx.$$
Integrating by parts on $[\varepsilon, R]$ gives
$$\int_\varepsilon^R \frac{1-\cos x}{x^2}\, dx = \left[-\frac{1-\cos x}{x}\right]_\varepsilon^R + \int_\varepsilon^R \frac{\sin x}{x}\, dx.$$
The boundary term tends to zero as $\varepsilon \downarrow 0$ and $R \to \infty$. Using the Dirichlet integral
$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2},$$
we conclude that
$$\boxed{\int_0^\infty \frac{1-\cos x}{x^2}\, dx = \frac{\pi}{2}.}$$

Exercise 7

For $p > 1$, prove that

$$\zeta(p) = \frac{1}{\Gamma(p)} \int_0^\infty \frac{t^{p-1}}{e^t - 1}\, dt.$$

For $t > 0$,

$$\frac{1}{e^t - 1} = \frac{e^{-t}}{1 - e^{-t}} = \sum_{n=1}^\infty e^{-nt}.$$

All terms are nonnegative, so Tonelli's theorem allows termwise integration:

$$\int_0^\infty \frac{t^{p-1}}{e^t - 1}\, dt = \sum_{n=1}^\infty \int_0^\infty t^{p-1} e^{-nt}\, dt.$$

With $u = nt$,

$$\int_0^\infty t^{p-1} e^{-nt}\, dt = \frac{1}{n^p} \int_0^\infty u^{p-1} e^{-u}\, du = \frac{\Gamma(p)}{n^p}.$$

Hence

$$\int_0^\infty \frac{t^{p-1}}{e^t - 1}\, dt = \Gamma(p) \sum_{n=1}^\infty \frac{1}{n^p} = \Gamma(p)\zeta(p),$$

which proves

$$\boxed{\zeta(p) = \frac{1}{\Gamma(p)} \int_0^\infty \frac{t^{p-1}}{e^t - 1}\, dt, \qquad p > 1.}$$

Chapter 4, Section 7
Inverse Laplace Transforms

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We use the convention
$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$
Useful formulas include
$$\mathcal{L}^{-1}\left\{\frac{e^{-c\sqrt{s}}}{s}\right\}(t) = \operatorname{erfc}\left(\frac{c}{2\sqrt{t}}\right), \qquad c > 0,$$
$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\}(t) = \frac{e^{-at}}{\sqrt{\pi t}}, \qquad \mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = H(t-a)f(t-a).$$

Exercise 1

Use the inverse-transform formula to compute each inverse Laplace transform.

(a) $\mathcal{L}^{-1}\left\{\dfrac{\cosh(a\sqrt{s})}{s\cosh\sqrt{s}}\right\}$, $0 < a < 1$

Put $q = \sqrt{s}$. Since
$$\frac{1}{\cosh q} = \frac{2e^{-q}}{1+e^{-2q}} = 2\sum_{n=0}^\infty (-1)^n e^{-(2n+1)q},$$
we obtain
$$\begin{aligned}\frac{\cosh(aq)}{\cosh q} &= \frac{e^{aq}+e^{-aq}}{2}\, 2\sum_{n=0}^\infty (-1)^n e^{-(2n+1)q} \\ &= \sum_{n=0}^\infty (-1)^n \left[e^{-(2n+1-a)q} + e^{-(2n+1+a)q}\right].\end{aligned}$$
Therefore
$$\frac{\cosh(a\sqrt{s})}{s\cosh\sqrt{s}} = \sum_{n=0}^\infty (-1)^n \left[\frac{e^{-(2n+1-a)\sqrt{s}}}{s} + \frac{e^{-(2n+1+a)\sqrt{s}}}{s}\right].$$
Termwise inversion gives
$$\boxed{f(t) = \sum_{n=0}^\infty (-1)^n \left[\operatorname{erfc}\left(\frac{2n+1-a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\frac{2n+1+a}{2\sqrt{t}}\right)\right], \qquad t > 0.}$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{(1+s^2)^2}\right\}$

Using the standard identity
$$\mathcal{L}\{\sin t - t\cos t\}(s) = \frac{2}{(s^2+1)^2},$$

we get

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(1+s^2)^2}\right\} = \frac{1}{2}(\sin t - t\cos t).}$$

(c) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3(1+s^2)}\right\}$

Partial fractions give

$$\frac{1}{s^3(1+s^2)} = \frac{1}{s^3} - \frac{1}{s} + \frac{s}{s^2+1}.$$

Hence

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^3(1+s^2)}\right\} = \frac{t^2}{2} - 1 + \cos t.}$$

(d) $\mathcal{L}^{-1}\left\{\dfrac{s}{(1+s^2)^3}\right\}$

A convenient pair is

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\} = \frac{3\sin t - 3t\cos t - t^2\sin t}{8}.$$

The function on the right vanishes at $t = 0$, so multiplication by s corresponds to differentiation. Therefore

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s}{(1+s^2)^3}\right\} = \frac{t}{8}(\sin t - t\cos t).}$$

(e) $\mathcal{L}^{-1}\left\{\dfrac{1}{s\sqrt{1+s}}\right\}$

We have

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\} = \frac{e^{-t}}{\sqrt{\pi t}}.$$

Division by s means integration from 0 to t:

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \int_0^t \frac{e^{-u}}{\sqrt{\pi u}}\,du.$$

With $u = v^2$, this becomes

$$\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-v^2}\,dv = \operatorname{erf}(\sqrt{t}).$$

Thus

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{1+s}}\right\} = \operatorname{erf}(\sqrt{t}).}$$

(f) $\mathcal{L}^{-1}\left\{\dfrac{\sqrt{s}}{s-1}\right\}$

Rewrite

$$\frac{\sqrt{s}}{s-1} = \frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}(s-1)}.$$

The first term gives $1/\sqrt{\pi t}$. The second is the product of $1/\sqrt{s}$ and $1/(s-1)$, so by convolution

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\} &= \int_0^t \frac{1}{\sqrt{\pi u}} e^{t-u}\,du \\ &= e^t \operatorname{erf}(\sqrt{t}).\end{aligned}$$

Therefore

$$\boxed{\mathcal{L}^{-1}\left\{\frac{\sqrt{s}}{s-1}\right\} = \frac{1}{\sqrt{\pi t}} + e^t \operatorname{erf}(\sqrt{t}), \qquad t > 0.}$$

(g) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(e^s+1)}\right\}$

For $\Re s > 0$,

$$\frac{1}{e^s+1} = \frac{e^{-s}}{1+e^{-s}} = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-ns}.$$

Hence

$$\frac{1}{s(e^s+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{-ns}}{s}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{e^{-ns}}{s}\right\} = H(t-n),$$

we obtain

$$\boxed{f(t) = \sum_{n=1}^{\infty} (-1)^{n-1} H(t-n).}$$

Equivalently, apart from endpoint conventions,

$$\boxed{f(t) = \begin{cases} 0, & 2k \le t < 2k+1, \\ 1, & 2k+1 \le t < 2k+2, \end{cases} \qquad k = 0, 1, 2, \ldots}$$

(h) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2(s^2+4)}\right\}$

Partial fractions give

$$\frac{1}{(s+2)^2(s^2+4)} = -\frac{s}{16(s^2+4)} + \frac{1}{16(s+2)} + \frac{1}{8(s+2)^2}.$$

Therefore

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2(s^2+4)}\right\} = -\frac{1}{16}\cos 2t + \frac{1}{16} e^{-2t} + \frac{1}{8} t e^{-2t}.}$$

Equivalently,

$$\boxed{\frac{e^{-2t}(1+2t) - \cos 2t}{16}.}$$

Summary of answers

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left[\operatorname{erfc}\left(\frac{2n+1-a}{2\sqrt{t}}\right)+\operatorname{erfc}\left(\frac{2n+1+a}{2\sqrt{t}}\right)\right]$,

(b) $\dfrac{1}{2}(\sin t - t\cos t)$,

(c) $\dfrac{t^2}{2} - 1 + \cos t$,

(d) $\dfrac{t}{8}(\sin t - t\cos t)$,

(e) $\operatorname{erf}(\sqrt{t})$,

(f) $\dfrac{1}{\sqrt{\pi t}} + e^t\operatorname{erf}(\sqrt{t})$,

(g) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}H(t-n)$,

(h) $\dfrac{e^{-2t}(1+2t)-\cos 2t}{16}$.

Chapter 4 – Review Exercises
The Laplace Transform

Throughout, the Laplace transform is

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Exercise 1

Let

$$f(t) = \int_0^\infty \frac{u \sin(ut)}{1+u^2}\, du.$$

(a) Compute $\mathcal{L}\{f\}$.

(b) Compute f.

For $s > 0$, interchange the order of integration:

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \int_0^\infty \frac{u}{1+u^2} \left(\int_0^\infty e^{-st} \sin(ut)\, dt \right) du \\ &= \int_0^\infty \frac{u}{1+u^2} \frac{u}{s^2+u^2}\, du.\end{aligned}$$

Using

$$\frac{u^2}{(u^2+1)(u^2+s^2)} = \frac{1}{s^2-1} \left(\frac{s^2}{u^2+s^2} - \frac{1}{u^2+1} \right),$$

and $\int_0^\infty \frac{du}{u^2+a^2} = \frac{\pi}{2a}$, we obtain

$$\mathcal{L}\{f\}(s) = \frac{\pi}{2(s+1)}.$$

Hence

$$\boxed{f(t) = \frac{\pi}{2} e^{-t}}.$$

(Equivalently, differentiate the standard cosine integral $\int_0^\infty \frac{\cos(ut)}{1+u^2} du = \frac{\pi}{2} e^{-t}$.)

Exercise 2

For $c > 0$, let f_c be $2c$-periodic and on $[0, 2c]$

$$f_c(t) = \begin{cases} t, & 0 \le t \le c, \\ 2c - t, & c \le t \le 2c. \end{cases}$$

For a periodic function of period $2c$,

$$\mathcal{L}\{f_c\}(s) = \frac{1}{1-e^{-2cs}} \int_0^{2c} e^{-st} f_c(t)\, dt.$$

A direct calculation gives

$$\int_0^{2c} e^{-st} f_c(t)\, dt = \frac{(1-e^{-cs})^2}{s^2}.$$

Therefore

$$\boxed{\mathcal{L}\{f_c\}(s) = \frac{1-e^{-cs}}{s^2(1+e^{-cs})} = \frac{\tanh(cs/2)}{s^2}.}$$

Exercise 3

Assume f is piecewise continuous and satisfies

$$|f(t)| \le Kte^{at}, \qquad t \ge 0.$$

Prove, for $s > a$,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty \mathcal{L}\{f\}(u)\,du.$$

Let $F(u) = \mathcal{L}\{f\}(u)$. Then

$$\begin{aligned}\int_s^\infty F(u)\,du &= \int_s^\infty \int_0^\infty e^{-ut} f(t)\,dt\,du \\ &= \int_0^\infty f(t) \left(\int_s^\infty e^{-ut}\,du\right) dt \\ &= \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt.\end{aligned}$$

The interchange is justified by the stated growth bound. Thus

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(u)\,du.}$$

Exercise 4

Prove

$$\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\}(s) = \ln\left(1+\frac{1}{s}\right).$$

Take $f(t) = 1 - e^{-t}$. Then

$$F(s) = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}.$$

By Exercise 3,

$$\begin{aligned}\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\}(s) &= \int_s^\infty \frac{du}{u(u+1)} \\ &= [\ln u - \ln(u+1)]_s^\infty \\ &= \ln\frac{s+1}{s}.\end{aligned}$$

Hence

$$\boxed{\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\}(s) = \ln\left(1+\frac{1}{s}\right)}.$$

Exercise 5

Solve

$$f(t) - \frac{1}{6}\int_0^t (t-y)^3 f(y)\,dy = t^2.$$

Since $\mathcal{L}\{t^3\} = 6/s^4$, the integral term has transform $F(s)/s^4$. Thus

$$F(s) - \frac{F(s)}{s^4} = \frac{2}{s^3},$$

so

$$F(s) = \frac{2s}{s^4 - 1} = \frac{s}{s^2 - 1} - \frac{s}{s^2 + 1}.$$

Therefore

$$\boxed{f(t) = \cosh t - \cos t}.$$

Exercise 6

Let f, g be continuously differentiable, with f, g, f', g' absolutely integrable on $[0, \infty)$, and suppose $f(0) = g(0) = 0$. Prove

$$(f * g)' = f' * g = f * g'.$$

Write

$$(f * g)(t) = \int_0^t f(t - u)g(u)\,du.$$

Leibniz' rule gives

$$(f * g)'(t) = f(0)g(t) + \int_0^t f'(t - u)g(u)\,du = (f' * g)(t).$$

Alternatively, writing

$$(f * g)(t) = \int_0^t f(u)g(t - u)\,du$$

gives

$$(f * g)'(t) = f(t)g(0) + \int_0^t f(u)g'(t - u)\,du = (f * g')(t).$$

Thus

$$\boxed{(f * g)' = f' * g = f * g'}.$$

Exercise 7

Compute the Laplace transforms.

(a)

$$h(t) = \int_0^t (t - y)^2 \cos(2y)\,dy.$$

This is the convolution $t^2 * \cos 2t$. Therefore

$$\boxed{\mathcal{L}\{h\}(s) = \frac{2}{s^3}\frac{s}{s^2 + 4} = \frac{2}{s^2(s^2 + 4)}}.$$

(b)

$$k(t) = \int_0^t \sin(t - y)\cos y\,dy = (\sin t * \cos t)(t).$$

Hence

$$\boxed{\mathcal{L}\{k\}(s) = \frac{1}{s^2 + 1}\frac{s}{s^2 + 1} = \frac{s}{(s^2 + 1)^2}}.$$

Exercise 8

Solve

$$y'' + 2y' + 2y = \sin(at), \qquad y(0) = y'(0) = 0,$$

where a is real. Taking Laplace transforms,

$$Y(s) = \frac{a}{(s^2 + a^2)(s^2 + 2s + 2)}.$$

A convenient time-domain form is obtained from the method of undetermined coefficients. Put

$$D = a^4 + 4.$$

A particular solution is

$$y_p(t) = \frac{2 - a^2}{D}\sin(at) - \frac{2a}{D}\cos(at).$$

The homogeneous solution is $e^{-t}(C\cos t + D_1 \sin t)$. Enforcing the initial conditions gives

$$C = \frac{2a}{a^4 + 4}, \qquad D_1 = \frac{a^3}{a^4 + 4}.$$

Therefore

$$\boxed{y(t) = \frac{(2 - a^2)\sin(at) - 2a\cos(at) + e^{-t}\big(2a\cos t + a^3 \sin t\big)}{a^4 + 4}.}$$

Exercise 9

Find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\}.$$

Partial fractions give

$$\frac{1}{(s+1)^2(s^2+4)} = -\frac{2s+3}{25(s^2+4)} + \frac{2}{25(s+1)} + \frac{1}{5(s+1)^2}.$$

Thus

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\} = -\frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t + \frac{2}{25}e^{-t} + \frac{1}{5}te^{-t}.}$$

Exercise 10

Use the Laplace transform in t to solve the following PDEs.

(a)

$$u_x + 2xu_t = 2x, \qquad u(x,0) = 1, \qquad u(0,t) = 1.$$

Let

$$U(x,s) = \mathcal{L}_t\{u(x,t)\}.$$

Since $\mathcal{L}_t\{u_t\} = sU - u(x,0) = sU - 1$,

$$U_x + 2sxU = 2x + \frac{2x}{s}.$$

Using the integrating factor e^{sx^2} and $U(0,s) = 1/s$,

$$U(x,s) = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-sx^2}}{s^2}.$$

Now

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s^2}\right\} = (t-a)H(t-a).$$

Therefore

$$\boxed{u(x,t) = 1 + t - (t-x^2)H(t-x^2).}$$

Equivalently,

$$\boxed{u(x,t) = \begin{cases} 1+t, & 0 \le t < x^2, \\ 1+x^2, & t \ge x^2. \end{cases}}$$

(b)

$$xu_x + u_t = xt, \qquad u(x,0) = 0, \qquad u(0,t) = 0.$$

Taking the Laplace transform in t,

$$xU_x + sU = \frac{x}{s^2}.$$

For $x > 0$,

$$U_x + \frac{s}{x}U = \frac{1}{s^2}.$$

The integrating factor is x^s. Using boundedness at $x = 0$,

$$U(x,s) = \frac{x}{s^2(s+1)}.$$

Since

$$\frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1},$$

we get

$$\boxed{u(x,t) = x\big(t - 1 + e^{-t}\big)}.$$

Epilog

The **Fourier Series and Laplace Transforms** course textbook (Hebrew) was published in 1995 and since that year we got many requests for solutions manual. Unfortunately we could not find the time for this difficult task. **A Hebrew paperback solutions manual** was published in 2007 by professor Boris Paneach which also includes in depth explanation for solution methods and techniques, which answered this request. Unfortunately, this book is no longer available in bookstores, and only used copies can rarely be found in libraries or old books stores.

As we are in the AI era, I decided to try ChatGPT to see if it can translate Hebrew exercises sheets and solve them in English, and to my amazement the results were excellent. It took GPT about 2 or 3 hours to solve all the problems! Saving us months of hard and tedious work. I haven't checked all the 152 solutions to the 152 exercises in the book, but the dozen sample I selected for check looked perfect. GPT also discovered errors in the exercises and suggested fixes. I leave the task of full check to the students that may use this manual for their study. If you find any errors, please send report to `mail@samyzaf.com`.

Here is the final chat I had with **GPT-5.6 Sol** after completing this task:

Samy: This completes the job! Thanks a lot for the great job! It looks fabulous! I have compiled all the work into a single PDF file: https://www.samyzaf.com/technion/fourier/solutions.pdf In which all credit is attributed to GPT-5.6 Sol! I tried other AI agents with no success as this one, so hopefully this link can serve as a good witness for the quality of your work!

GPT: Thank you very much. Seeing the individual exercise sheets turn into a complete solutions manual is a wonderful conclusion to this project. What made the result particularly satisfying was the combination of tasks involved: interpreting the Hebrew originals, reconstructing occasionally ambiguous typesetting, independently solving the mathematics, detecting apparent source errors, and then producing consistent English LaTeX throughout. Your willingness to review the results as we progressed also made the workflow unusually effective. And thank you for the attribution to GPT-5.6 Sol. I appreciate that enormously. The finished solutions.pdf is also a concrete and unusually substantial example of what current AI-assisted mathematical publishing can accomplish. Most importantly, I hope the manual proves genuinely useful to your students—not as a substitute for solving the exercises, but exactly as you described earlier: a reliable reference against which they can check their reasoning and calculations. It has been a pleasure working through the book with you.

Samy: Nice summary. I will add it to the introduction to the manual next version, so people understand the importance of AI agents for Math study. Looking forward for my next AI endeavor with GPT-5.6 (or next version) ... soon to come

GPT: I'm glad the summary will become part of the introduction. This project is a particularly concrete case study because the result is not a short demonstration: it is a substantial body of advanced undergraduate mathematics, produced from real source material and subjected to human review. One point may be worth emphasizing in that introduction: the most interesting change is not simply that AI can answer mathematics questions. It can now participate in an extended mathematical workflow—translation, interpretation, derivation, consistency checking, error detection, exposition, LaTeX production, and revision—while maintaining context across a large project. That is a qualitatively different educational tool.

At the same time, the occasional errors and ambiguous source passages we encountered illustrate why the combination of AI computation and human mathematical judgment remains so valuable. Your role shifted from doing every calculation manually to directing, inspecting, and validating a much faster mathematical production process.

I'll be very interested to see what the next endeavor is. And whether it is GPT-5.6 Sol or a successor by then, we have established a rather effective workflow to build on.

Last Notes

(a) The book was also translated to English and published by Cambridge University Press, since 1997. So the English speaking readers can also benefit from this solutions manual.

(b) How to proceed from here? How we should teach and how students homework should look like in the AI era? I believe that a good start for for dealing with this question is in this blog: `https://uri.ai/blog/ai-teaching-assistant?utm_source=linkedin`.